Some fun with Euler’s identity

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EDIT I don’t know why I thought computing a series expansion for e^{-1} using Euler’s identity was clever, while it’s just way simpler via the usual expansion of e^x in the real numbers… I’m a bit ashamed of this post.

The classic identity $e^{ix} = \cos x + i \sin x$ works for all complex values of x. Hence, by setting $x = i$ we find

$$e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots.$$

Therefore, we have

$$1 = \left(2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right)\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots\right).$$

Distributing yields

$$1 - 2e^{-1} = \sum_{k,\ell \geq 2} \frac{(-1)^k}{k!\ell!}.$$

Notice that in the previous sum, each pair (k, ℓ) where k is odd and ℓ is even cancels out the term corresponding to the pair (ℓ, k). Hence we may sum over k and ℓ that have the same parity:

$$1 - 2e^{-1} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{(2m)!(2n)!} - \frac{1}{(2m+1)!(2n+1)!}\right).$$