Revisiting the infinitude of primes

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Theorem. The prime numbers are inexhaustible.

Let's cast this as a geometrical problem. We will study the scheme $\operatorname{Spec}\mathbb{Z}$. The points of this scheme are (identified with) the prime ideals of $\mathbb{Z}$, and it is a fun exercise to prove that they are precisely: the zero ideal (0); and principal ideals (p) where p is a prime number in $\mathbb{N}$ (hint: simply Euclid's Lemma).

Suppose for a contradiction that there are only a finite number of prime numbers. Let m be the product of all primes. Then, the integer $m+1$ can be interpreted as a regular function over $\operatorname{Spec}\mathbb{Z}$ that vanishes nowhere. Any such function is necessarily invertible, but the only units in $\mathbb{Z}$ are ± 1. Hence, either $m = 0$ or $m = -2$, two impossibilities. ■

Discussion. The key fact is that any nowhere vanishing function is invertible. The reason this is true is because the ideal generated by a non-unit element is always contained in a maximal ideal. In our case, working with $\mathbb{Z}$, this is just saying that every integer has a prime dividing it.