An Introduction to Differential Geometry and Curvature

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Section 1: Geometry of $curves$ in $\mathbb{R}^3$

Consider $\alpha(t) : [a, b] \to \mathbb{R}^3$ to be a differentiable, time-parameterized curve through $\mathbb{R}^3$. The length of such a curve is defined as

$$L(\alpha) := \int_a^b \|\alpha'(t)\|_2 \, dt. \tag{1}$$

We say α is *parameterized by arc-length (p.a.l.)* if $\|\alpha'(t)\|_2 = 1$ for all $t \in [a, b]$. Now we'll introduce some definitions to build up to a notion of curvature. Supposing $\alpha : [a, b] \to \mathbb{R}^3$ p.a.l., we define

$$T(t) := \alpha'(t) \tag{2}$$

and call it the *unit tangent vector* to α at time t. Intuitively, we define the *curvature of the curve* α, denoted $k(t)$ to be the magnitude of the time derivative of $T(t)$,

$$k(t) := \|\alpha''(t)\|_2 = \|T'(t)\|_2. \tag{3}$$

Put simply, the curvature tells us how quickly the unit tangent vector of our curve changes! A little more digging will give allow us to draw a clear picture of all of these quantities. Notice that, since $\|T(t)\|_2 = 1$, we have

$$\begin{aligned} \frac{d}{dt}\langle T(t), T(t)\rangle &= 0 \\ \langle T'(t), T(t)\rangle + \langle T(t), T'(t)\rangle &= 0 \\ \langle T'(t), T(t)\rangle &= 0. \end{aligned}$$

This tells us that $T'(t)$ is necessarily orthogonal to the unit tangent, $T(t)$. We can use this insight to further define the *unit normal vector*,

$$n(t) := \frac{T'(t)}{\|T'(t)\|_2} = \frac{T'(t)}{k(t)}. \tag{4}$$

This gives us a solid basis of geometric descriptors of a given differentiable curve α. A depiction of an example $\alpha(t)$, $T(t)$ and $n(t)$ is shown HERE!

Section 2: Geometry of *surfaces* in $\mathbb{R}^3$

Now we can graduate to analyzing the geometry of *surfaces* in $\mathbb{R}^3$. Before we do so, though, we will define the *differential* of a map $X : \mathbb{R}^2 \to \mathbb{R}^3$. The differential of X at p, denoted $dX_p : \mathbb{R}^2 \to \mathbb{R}^3$, is a linear map that takes tangent vectors to curves passing through p to the associated tangent vector of the curve induced by X, show in Figure HERE. Formally, let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a curve such that $\alpha(0) = p$ and $\alpha'(0) = v$. Then,

$$dX_p(v) = \frac{d}{dt}(X \circ \alpha)\Big|_{t=0}. \tag{5}$$

While this definition may seem confusing, if we take a look at the matrix representation of this linear map, we see that it simply boils down to the jacobian of X applied to the vector v,

$$dX_p(v) = J_X v. \tag{6}$$

Remark The function $dX_p(v)$ depends only on p and v, not the choice of $\alpha(t) = (u(t), v(t))$.

This becomes clear when we expand the definition of the differential using the chain rule,

$$\begin{aligned} dX_p(v) &= \frac{d}{dt}(X \circ \alpha)\Big|_{t=0} \\ &= X_u u'(t) + X_v v'(t)\Big|_{t=0} \\ &= X_u u'(0) + X_v v'(0) \end{aligned}$$

where $v = (u'(0), v'(0))$ by definition and $X_u = \frac{\partial X}{\partial u}$, $X_v = \frac{\partial X}{\partial v}$.

Now we can finally discuss surfaces in $\mathbb{R}^3$, specifically *regular* surfaces.

Definition
$S \subset \mathbb{R}^3$ is called a *regular surface* if $\forall p \in S$, $\exists$ an open subset V of $\mathbb{R}^3$ such that $p \in V$ and there exists a surjective, C^∞ map $X : U \subset \mathbb{R}^2 \to V \cap S$ such that

1. X is a homeomorphism
2. dX_p is injective $\forall p \in U$.

X is often called a *local parameterization* or a *local chart* of S. This definition establishes the notion that a regular surface $S \subset \mathbb{R}^3$ can be constructed by stitching together deformations of patches of $\mathbb{R}^2$. Then, by analyzing these deformations we'll be able to quantify and characterize the geometry of S!

Tangent curves and spaces

We'll now define the notion of a tangent space to a point, a critical geometric object in Differential Geometry. Given a regular surface $S \subset \mathbb{R}^3$, we we say $v \in \mathbb{R}^3$ is a tangent vector if there exists a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. A visualization is provided HERE!

Definition The tangent space to S at p is defined as

$$T_pS = \{v \in \mathbb{R}^3 \,|\, v \text{ tangential to } S \text{ at } p\}.$$

Figure HERE! shows an example of the tangent space to a surface. This vector space T_pS actually arises directly from the charts used to construct S. This is illustrated in the following lemma:

Lemma Let $X : U \to S$ be a local chart around p. Then $T_pS = dX_{X^{-1}(p)}(\mathbb{R}^2)$.

I've omitted the proof, but it arises quite simply from the definitions of the differential and the tangent space. A simple corollary of this result follows too,

Corollary Let $X : U \to S$ be a local chart around p. Then $T_pS = \text{span}\{X_u, X_v\}$.

Critically, this means we have access to a basis of our tangent space about each $p \in S$ arising only from the chart X. Now that we've established a notion of tangency, we'll do the same for normalcy.

Normal vector fields and the Gauss map

A vector field on S (i.e. a differentiable map $V : S \to \mathbb{R}^3$) is called $normal$ if $\forall p \in S$ we have $\langle V(p), w\rangle = 0 \ \forall w \in T_pS$. Furthermore, V is called $unitary$ if $\|V(p)\|_2 = 1 \, \forall p$.

Unitary normal vector fields give rise to the notion of $orientability$. For example, non-orientable surfaces like the Möbius strip do not admit a global unitary normal vector field. This idea is beatifully illustrated by a painting by M.C. Escher, shown HERE!

If it exists, the unitary normal vector field $N : S \to \mathbb{R}^3$ for a surface S is called the $Gauss\ map$. Studying the behavior of the Gauss map will lead to a generalization of the notion of curvature for $surfaces$.

The First and Second Fundamental Forms

The first fundamental form (denoted I_p) and second fundamental form (denoted II_p) are quadratic forms on the tangent space of surfaces - namely, they take a vector pair $(v, w) \in T_pS \times T_pS$ to a number in $\mathbb{R}$. These two maps describe describe respectively the *intrinsic* and *extrinsic* geometry of a surface S, and will be the subject of our study for the rest of the blog post.

Definition Suppose we have a surface S with a local chart $X : U \subset \mathbb{R}^2 \to S$, and a point $p \in X(U)$. The *first fundamental form* of S at p, denoted $\mathrm{I}_p : T_pS \times T_pS \to \mathbb{R}$, is the map

$$\mathrm{I}_p(v, w) = \langle dX_p(v), dX_p(w) \rangle \tag{7}$$

for $(v, w) \in U \times U$. It is often interpreted as the restriction of the Euclidean inner product to T_pS.

A visualization of the quantities used in the definition can be seen in HERE! For me, it is intuitive to think about the matrix representation of I_p. Expanding the definition we see,

$$\begin{aligned} \mathrm{I}_p(v, w) &= \langle dX_p(v), dX_p(w) \rangle \\ &= v^T J_X^T J_X w \\ &= v^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} w \end{aligned}$$

where

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{bmatrix}.$$

By looking at the first fundamental form this way, we see that it tells us how much the local chart X deforms the patch $U \subset \mathbb{R}^2$ along the coordinate axes u and v to get the patch $X(U) \subset S$! We can make this idea concrete by computing path lengths on S, where we will see a clear dependence on the first fundamental form.

Let $\beta : [a, b] \to U \subset \mathbb{R}^2$ and let $\alpha = X \circ \beta$, where X is a local chart for S. Lets compute the length of the path of β and α:

$$L(\beta) = \int_a^b \|\beta'(t)\|_2 \, dt$$
$$L(\alpha) = \int_a^b \|\alpha'(t)\|_2 \, dt.$$

We can apply the chain rule to get $\alpha'(t)$ in terms of β,

$$\begin{aligned}\alpha'(t) &= \frac{d}{dt}(X \circ \beta)(t) \\ &= X_u u'(t) + X_v v'(t).\end{aligned}$$

Plugging back into the path length equation, we get

$$L(\alpha) = \int_a^b \Big\langle X_u u'(t) + X_v v'(t),\, X_u u'(t) + X_v v'(t) \Big\rangle^{1/2} dt$$
$$L(\alpha) = \int_a^b \Big(E u'(t)^2 + 2F u'(t) v'(t) + G v'(t)^2 \Big)^{1/2} dt.$$

Rewriting $L(\beta)$ as follows,

$$L(\beta) = \int_a^b \Big(u'(t)^2 + v'(t)^2 \Big)^{1/2} dt$$

clearly illustrates the fact that I_p determines the distortion of path lengths induced by the chart X. Now, we will move on to define and analyze the $second$ fundamental form.

Definition Suppose we have a surface S with a local chart $X : U \subset \mathbb{R}^2 \to S$, and a point $p \in X(U)$. The $second\ fundamental\ form$ of S at p, denoted $\mathrm{I\!I}_p : T_pS \times T_pS \to \mathbb{R}$, is the map

$$\mathrm{I\!I}_p(v, w) = \langle -dN_p(v), w \rangle \tag{8}$$

for $(v, w) \in U \times U$.

Again, we can look at the matrix representation of $\mathrm{I\!I}_p$ to get a better idea of what is going on,

$$\begin{aligned}\mathrm{I\!I}_p(v, w) &= \langle -dN_p(v), w \rangle \\ &= v^T(-J_N)w\end{aligned}$$

where J_N is the jacobian of the Gauss map. Some calculation reveals that

$$J_N = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

where,

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} -\langle N_u, X_u \rangle & -\langle N_v, X_u \rangle \\ -\langle N_u, X_v \rangle & -\langle N_v, X_v \rangle \end{bmatrix}.$$

Again, this sheds some light on what $\mathrm{I\!I}_p$ represents: it tells us how the normal vector changes as we move along the surface (specifically with respect to changes in the parameters u and v in the domain of the local chart X).

Theorem (Euler) For a surface S suppose we have a unitary normal vector field $N : S \to \mathbb{R}^3$. Also suppose that we have $\alpha : (-\epsilon, \epsilon) \to S$ p.a.l., such that $\alpha(0) = p$, $\alpha'(0) = v$. Then

$$\mathrm{I\!I}_p(v, v) = \langle \alpha''(0), N(p) \rangle. \tag{9}$$

The term on the right, $\langle \alpha''(0), N(p) \rangle$, represents the projection of the acceleration of α onto the surface normal. This quantity is often referred to as the $normal\ curvature$, as it represents the amount of curvature of α that is $normal$ to the surface S at p.

Proof of Euler's Theorem:

$$\begin{aligned} \mathrm{I\!I}_p(v, v) &= \mathrm{I\!I}_p(\alpha'(0), \alpha'(0)) \\ &= \langle dN_p(\alpha'(0)), \alpha'(0) \rangle \\ &= \Big\langle \underbrace{\frac{d}{dt} N(\alpha(t))\Big|_{t=0}}_{N'(p)}, \alpha'(0) \Big\rangle \\ &= \Big\langle N(p), \alpha''(0) \Big\rangle \end{aligned}$$

where the last step follows from the fact that $N(\alpha(t)) \perp \alpha'(t)$.

Definition The $principal\ curvatures$ of a surface S at p are defined to be the $maximum$ and $minimum$ normal curvatures that an arc-length parameterized curve α can have, where $\alpha(0) = p$. Formally,

$$\begin{aligned} k_1(p) &= \min_{\alpha(0)=p} k_n(0) \\ k_2(p) &= \max_{\alpha(0)=p} k_n(0). \end{aligned}$$

From the principal curvatures, the definitions of $Gaussian$ and $mean$ curvature arise:

Definition The $Gaussian\ curvature$ of a surface S at p is the product of the principal curvatures,

$$K(p) = k_1(p)k_2(p) \tag{10}$$

while the $mean$ curvature is the mean of the principal curvatures,

$$H(p) = \frac{k_1(p) + k_2(p)}{2}. \tag{11}$$

For the rest of the blog we'll focus on $Gaussian$ curvature. While mean curvature offers a tool for the study of the extrinsic geometry of a surface, it is out of the scope of this blog post.

To give some grounding for Gaussian curvature: the plane has $K \equiv 0$, the standard 2-sphere has $K \equiv +1$, while the standard pseudosphere has $K \equiv -1$. But a key question arises: $how\ were\ those\ quantities\ computed?$ It's not clear how to solve the minimization problem required to compute the principal curvatures. But if we study $\mathrm{I\!I}_p$ a little more, we'll find that it has the ingredients to allow us to compute k_1 and k_2 directly from a local chart X! Recall

$$\begin{aligned} k_1(p) &= \min_{\alpha(0)=p} k_n(0) \\ &= \min_{\alpha(0)=p} \Big\langle N(p), \alpha''(0) \Big\rangle. \end{aligned}$$

We can directly apply Euler's Theorem to replace the right hand side with the second fundamental form applied to the initial velocity of α,

$$\begin{aligned} k_1(p) &= \min_{\alpha(0)=p} \Big\langle N(p), \alpha''(0) \Big\rangle \\ &= \min_{\alpha(0)=p} \mathrm{I\!I}_p(\alpha'(0), \alpha'(0)). \end{aligned}$$

Because this last line only depends on the zero and first-order behavior of α, we can replace it with the following,

$$\begin{aligned} k_1(p) &= \min_{\|v\|_2=1} \mathrm{I\!I}_p(v, v) \\ &= \min_{\|v\|_2=1} v^T(-J_N)v. \end{aligned}$$

The interpretations of the eigenvalues of a quadratic form as the minimal and maximal values of the quadratic form restricted to the unit sphere tells us that the eigenvalues λ_1, λ_2 of $-J_N$ are the principal curvatures:

$$k_1(p) = \lambda_1$$
$$k_2(p) = \lambda_2.$$

This gives rise to new equations for the Gaussian and mean curvatures:

$$K(p) = \det(-J_N) = \frac{eg - f^2}{EG - F^2}$$
$$H(p) = \frac{1}{2}\mathrm{Tr}(-J_N) = \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}.$$

Section 3: Gauss' Theorema Egregium

In this section we'll walk through Gauss' *Theorema Egregium*, one of the major results of differential geometry. To do so, we first need to establish the definition of *isometric maps* and some properties of them.

Definition Surfaces S, $\tilde{S} \subset \mathbb{R}^3$ are called *isometric* if there exists a diffeomorphism (a C^1 map with a C^1 inverse) $f : S \to \tilde{S}$ that takes each curve on S to a curve of the same length on $\tilde{S}$.

An example of an isometric surface pair, the plane and the S-surface, is shown in HERE! These surfaces are isometric because we can bend, but not stretch, the plane so that it coincides with the S-surface. Such a transformation does not distort the length of paths!

Theorem Patches of surfaces S, $\tilde{S} \subset \mathbb{R}^3$ are *isometric* if there exist charts $X : U \subset \mathbb{R}^2 \to S$ and $\tilde{X} : U \subset \mathbb{R}^2 \to \tilde{S}$ such that the first fundamental form I_p coincides.

Proof sketch: An isometric map between patches of S and $\tilde{S}$ preserves path lengths. We also have shown that the distortion of such path lengths incurred by the charts X and $\tilde{X}$ is determined completely by their respective first fundamental forms. From here, it's not hard to show that since an isometry exists (and path lengths are therefore preserved under this map) the first fundamental forms of the two surfaces must coincide. ADD FIGURE HERE!

Now we are in a position where we can state Gauss' *Theorema Egregium*.

Theorem (Gauss) The Gaussian curvature of a surface S is *invariant* under isometries. That is, if $f : S \to S'$ is an isometry, then $K(p) = K(f(p))\ \forall p \in S$.

Proof sketch. One can show that the Gaussian curvature K depends only on the zero and first order behavior of the first fundamental form E, F and G. Thus, K can be written as

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{\phi(E, G, F, E_u, E_v, G_u, G_v, F_u, F_v)}{EG - F^2} \end{aligned}$$

where ϕ is some messy function of the first fundamental form and its partial derivatives w.r.t. u and v. Since K can be written as a function of only the first fundamental form, and by the previous theorem isometries preserve the first fundamental form, we can conclude that the Gaussian curvature K must be preserved under isometries!

Implications

Gauss' $Theorema\ Egregium$ is a major result of differential geometry, as it sheds light on the distinction between $intrinsic$ and $extrinsic$ curvature. Take, for example, an open subset of the plane and roll it into a cylinder. Intuitively (and one can check this by computing E, F, G) the lengths of paths are completely unchanged. Since $K \equiv 0$ for the plane, Gauss' $Theorema\ Egregium$ tells us that this must mean $K \equiv 0$ for the cylinder!

This arises from the fact that Gaussian curvature measures the $intrinsic$ distortion of a surface from a Euclidean space. Rolling the plane up into a cylinder introduces $extrinsic$ curvature, but does not stretch or warp the surface in any way. One can see this by playing around with a piece of printer paper. Anything you do to bend or fold the surface that does not introduce any tears or pleats is introducing $extrinsic$ curvature (and assuming no hard creases are made, such a transformation is an isometry)!

Another huge implication of the $Theorema\ Egregium$ is in the field of Cartography. Since $K \equiv 0$ for the plane and $K \equiv +1$ for the sphere, the surfaces must not be isometric. Therefore, one cannot create a $distance\text{-}preserving$ projection of the sphere onto the plane.

Bibliography

Reference