The Integrals of a Sequence of Iteratively Defined Functions.

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Whenever we get a break from working with clients, we like to help people with their matthematics and statistics problem on Reddit. The following problem was one that was posed this past weekend on Reddit.

Define $f^{(k)} \,|[0,1] \to [0,1]$, $k = 0, 1, \ldots$ iteratively as follows ,

$$f^{(0)}(x) := 2x(1-x)$$

and for $k \geq 1$

$$f^{(k)}(x) := f \circ f^{(k-1)}(x).$$

The problem is to evaluate

$$I_k := \int_0^1 f^{(k)}(x)\,\mathrm{d}x,$$

$k = 0, 1 \ldots$

To evaluate the integral, the substitution $\eta := x - \frac{1}{2}$ is made. With $g^{(k)}(\eta)\,\big|\left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined by

$$g^{(k)}(x) := f^{(k)}\left(\eta + \frac{1}{2}\right),$$

one has

$$I_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{(k)}(\eta)\,\mathrm{d}\eta$$

$k = 0, 1, \ldots$ The advantage conferred by this substitution is that the functions $g^{(k)}$ take on much simplier form that the functions $f^{(k)}$.

Note that

$$g^{(0)}(\eta) = 2\left[\eta + \frac{1}{2}\right]\left[1 - \left(\eta + \frac{1}{2}\right)\right] = 2\left[\frac{1}{2} + \eta\right]\left[\frac{1}{2} - \eta\right] = \frac{1}{2} - 2\eta^2,$$

and for $k \geq 1$

$$g^{(k)}(\eta) = f^{(k)}\left(\eta + \frac{1}{2}\right) = f\left(f^{(k-1)}\left(\eta + \frac{1}{2}\right)\right) = f \circ g^{(k-1)}(\eta).$$

It follows that

$$g^{(1)}(\eta) = 2\left[\frac{1}{2} - 2\eta^2\right]\left[\frac{1}{2} + 2\eta^2\right] = \frac{1}{2} - 2^3\eta^{2^2},$$

and

$$g^{(2)}(\eta) = 2\left[\frac{1}{2} - 2^3\eta^{2^2}\right]\left[\frac{1}{2} + 2^3\eta^{2^2}\right] = \frac{1}{2} - 2^7\eta^{2^3}.$$

At this point the pattern is clear, and induction can be used to prove that for $k \geq 0$

$$g^{(k)}(\eta) = \frac{1}{2} - 2^{2^{k+1}-1}\eta^{2^{k+1}}.$$

A straightforward integration yields

$$I_k = \frac{1}{2}\left[1 - \frac{1}{2^{k+1}+1}\right].$$