An Elementary Number Theory Problem

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A couple of days ago, the following question was posted on the MathHelp subreddit.

> **6 is divisible by all integers up to its half: 1, 2, and 3. Are there any numbers >6 with this property?**
>
> In other words, cases where x is divisible by all integers from 1 to x/2.
>
> Something tells me 6 is the highest but I have no idea how to go about proving it.

I gave the following response.

> Let me rephrase your question , as I understand it, more precisely. Does there exist a natural numbers $n > 6$ such that for all $k \leq \frac{n}{2}$ $k|n$ and for all $\frac{n}{2} < m \neq n$, $m \not| n$.? Did I get that correct?

and suggestion

> BTW, it should be straightforward to prove that no such natural number exists, by considering the prime decomposition of n.

In this post I will give a precise statement of the result and a full proof of it.

Proposition 1. *The only natural numbers n such that k is a divisor of n for all $k \leq \left\lfloor \frac{n}{2} \right\rfloor$ are $2, 4,$ and 6.*

Proof. It is easy to verify by inspection that $2, 4$, and 6 have this property while $1, 3$, and 5 do not. It is shown that 4 and 6 are the only numbers greater than or equal to 4 with this property by consideration of the prime decomposition of n

Assume that n has the property. For $n \geq 4$, $2 \leq \left\lfloor \frac{n}{2} \right\rfloor$. Hence $2|n$ and the prime decomposition of n takes the form

$$n = 2^{r_1} p_2^{r_2} \cdots p_l^{r_l}, \tag{1}$$

$r_1 \geq 1$,and $r_i \geq 0, i = 2, 3 \ldots, l$. It is assumed without loss of generality that $2 < p_2 < \cdots < p_l$. Clearly,

$$\frac{n}{2} = 2^{r_1-1}p_2^{r_2}\cdots p^{r_l}|n \tag{2}$$

Observe that $(\frac{n}{2}-1) < n$. Thus by assumption $(\frac{n}{2}-1)|n$. It follows that the prime decomposition of $\frac{n}{2}-1$ takes the form

$$\frac{n}{2} - 1 = 2^{s_1}p_2^{s_2}\cdots p_l^{s_l}, \tag{3}$$

where $s_i \leq r_i, i = 1, 2 \ldots, l$. Substitution from $(2)$and(3) followed by a factorization gives

$$1 = \frac{n}{2} - (\frac{n}{2} - 1) = p_2^{s_2}\cdots p^{s_l}\left[2^{r_1-1}p_2^{r_2-s_2}\cdots p_l^{r_l-s_l} - 2^{s_1}\right]. \tag{4}$$

If the product of two natural numbers equals 1, then each of the numbers equals 1. It follows that

$$p_2^{s_2}\cdots p^{s_l} = 1 \tag{5}$$

and

$$2^{r_1-1}p_2^{r_2-s_2}\cdots p_l^{r_l-s_l} - 2^{s_1} = 1. \tag{6}$$

From equation (5) , one obtains $s_i = 0, i = 2, 3, \ldots, l$. Thus equation (3) can be written as

$$\frac{n}{2} = 2^{s_1} + 1 \tag{7}$$

and equation (6) becomes

$$2^{r_1-1}p_2^{r_2}\cdots p_l^{r_l} = 2^{s_1} + 1 \tag{8}$$

If $s_1 = 0$, then (8) gives $r_1 = 2$ and $r_i = 0, i = 2, 3, \ldots, l$. In this case

$$n = 4. \tag{9}$$

On the other hand, suppose that $s_1 \geq 1$. Then consideration of (8) gives $r_1 = 1$. However, $s_1 \leq r_1$, consequently $s_1 = 1$. Again, consideration of (8) yields

$$p_2^{r_2}\cdots p_l^{r_l} = 3, \tag{10}$$

Therefore $p_2 = 3, r_2 = 1$, and $r_i = 0, i = 3, 4, \ldots, l$. Thus in this case

$$n = 6 \tag{11}$$

This concludes the proof. □