The Expected number of Transitions in n flips of a Coin

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This is a continuation of our test of the platform. Before making this post we uploaded the content of our custom LaTeX style file to the preamble. So far it is working fine.

Consider n independent flips of a coin with probability p of landing heads. A transition occurs whenever the outcome on a flip differs from the flip preceding it. The expected number of transition is found.

The outcomes of the flips are represented by an i.i.d sequence of Bernoulli(p) random variables $Y_1, Y_2, \ldots, Y_n$. The number of transition X is expressed as function of this sequence as follows. Define a new sequence of random variables in term of the $Y's$.

$$X_i = Y_i(1 - Y_{i-1}) + (1 - Y_i)Y_{i-1}, \tag{1}$$

$i = 2, \ldots, n$. Clearly, the sequence $X_2, X_3, \ldots, X_n$ is Bernoulli. Further, $X_i = 1$ if and only if a transition occurs on flip $i = 2, 3, \ldots, n$. Thus

$$X = X_2 + X_3 + \cdots + X_n. \tag{2}$$

Next,the expectation of X_i is found with the help of linearity of expectation, the independence of the $Y's$, and the fact that $\mathbf{E}[Y_i] = p$ as follows

$$\begin{aligned} \mathbf{E}[X_i] =& \mathbf{E}[Y_i(1 - Y_{i-1})] + \mathbf{E}[(1 - Y_i)Y_{i-1}] \\ =& \mathbf{E}[Y_i]\,\mathbf{E}[1 - Y_{i-1}] + \mathbf{E}[1 - Y_i]\,\mathbf{E}[Y_{i-1}] = pq + qp = 2pq, \end{aligned} \tag{3}$$

where $q := 1 - p$, $i = 2, 3, \ldots, n$. Thus the sequence is identically distributed Bernoulli($2pq$).

It follow from the linearity of expectation and the fact that $\mathbf{E}[X_i] = 2pq, i = 2, 3, \ldots, n$, that

$$\mathbf{E}[X] = 2(n - 1)pq. \tag{4}$$