A Test of the Platform

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This post is primarily intended as a test of the platform.

The indefinite integral

$$I := \int \frac{x^4}{x^4+1}\,\mathrm{d}x \tag{1}$$

is evaluated. First, observe that the integrand is an improper rational function, since the degree of the polynomial in the numerator is the same as the polynomial in the denominator. In such a situation, the first step is to express the improper rational function as the sum of a polynomial and a proper rational function, which can be achieved by long division. In this case, the improper rational function is simple enough that one may see by inspection that

$$\frac{x^4}{x^4+1} = 1 - \frac{1}{x^4+1}, \tag{2}$$

which may be verified by getting a common denominator on the right hand side.

The linearity of the indefinite integral along with (2) is used to write (1) as

$$I = \int \mathrm{d}x - \int \frac{1}{x^4+1}\,\mathrm{d}x = x - \int \frac{1}{x^4+1}\,\mathrm{d}x. \tag{3}$$

The first integral in the middle, which is straightforward to evaluate, has been evaluated. The second integral on the right is amenable to standard techniques of integration, however, it will take a few more steps to complete.

The standard technique to evaluate the integral of a proper rational function is partial fractions. This requires the factorization of the denominator into a product of linear polynomials and irreducible quadratic polynomials. Clearly, x^4+1 has no real roots, consequently it will only have irreducible quadratics polynomial in its factorization. One may verify by multiplying out the right hand side that

$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1). \tag{4}$$

Hence

$$\int \frac{\mathrm{d}x}{x^4+1} = \int \frac{\mathrm{d}x}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}. \tag{5}$$

Now one is in a position to employ partial fractions. Write

$$\frac{1}{x^4+1} = \frac{Ax+B}{(x^2+\sqrt{2}x+1)} + \frac{Cx+D}{(x^2-\sqrt{2}x+1)}. \tag{6}$$

A common denominator is found on the right and the resultant numerator on the right is set equal to the numerator on the left to obtain

$$(Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) = 1. \tag{7}$$

The left hand side is expanded and like powers are collected to give

$$(A+C)x^3+(B+D-\sqrt{2}A+\sqrt{2}C)x^2+(A+C-\sqrt{2}B+\sqrt{2}D)x+(B+D)=1. \quad (8)$$

The coefficients of like powers on the left and right are set equal, which yields the following set of simultaneous linear equation for the unknown coefficients

$$A+C=0 \quad (9)$$

$$B+D-\sqrt{2}A+\sqrt{2}C=0 \quad (10)$$

$$A+C-\sqrt{2}B+\sqrt{2}D=0 \quad (11)$$

$$B+D=1. \quad (12)$$

The first equation gives $C=-A$ and with a substitution into the third equation $B=D$. The fact that $B=D$ is used in the fourth equation to get $B=D=\frac{1}{2}$. Finally, the second equation gives $A=\frac{1}{2\sqrt{2}}=-C$. Consequently

$$\frac{1}{x^4+1}=\frac{1}{2\sqrt{2}}\left[\frac{x+\sqrt{2}}{(x^2+\sqrt{2}x+1)}-\frac{x-\sqrt{2}}{(x^2-\sqrt{2}x+1)}\right]. \quad (13)$$

Hence

$$\int\frac{\mathrm{d}x}{x^4+1}=\frac{1}{2\sqrt{2}}\left[\int\frac{x+\sqrt{2}}{(x^2+\sqrt{2}x+1)}\,\mathrm{d}x-\int\frac{x-\sqrt{2}}{(x^2-\sqrt{2}x+1)}\,\mathrm{d}x\right] \quad (14)$$

Next, the squares are completed in the denominators of the two integrands

$$\begin{aligned}(x^2+\sqrt{2}x+1)&=\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\\&=\frac{1}{2}\left[\left(\sqrt{2}x+1\right)^2+1\right]\end{aligned} \quad (15)$$

and

$$\begin{aligned}(x^2-\sqrt{2}x+1)&=\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\\&=\frac{1}{2}\left[\left(\sqrt{2}x-1\right)^2+1\right]\end{aligned} \quad (16)$$

Thus

$$\int\frac{\mathrm{d}x}{x^4+1}=\frac{1}{\sqrt{2}}\left[\int\frac{x+\sqrt{2}}{\left(\sqrt{2}x+1\right)^2+1}\,\mathrm{d}x-\int\frac{x-\sqrt{2}}{\left(\sqrt{2}x-1\right)^2+1}\,\mathrm{d}x\right] \quad (17)$$

The substitutions $\eta=\sqrt{2}x+1$ and $\xi=\sqrt{2}x-1$ are used to evaluate the two integrals on the right. To this end, observe that $\mathrm{d}\eta=\sqrt{2}\mathrm{d}x=\mathrm{d}\xi$, $\frac{\xi+1}{\sqrt{2}}=x=$

$\frac{\eta-1}{\sqrt{2}}$. Hence

$$\begin{aligned}\int \frac{x+\sqrt{2}}{\left(\sqrt{2}x+1\right)^2+1}\,\mathrm{d}x =&\frac{1}{\sqrt{2}}\int \frac{\frac{\eta-1}{\sqrt{2}}+\sqrt{2}}{\eta^2+1}\,\mathrm{d}\eta\\ =&\frac{1}{2}\int \frac{\eta+1}{\eta^2+1}\,\mathrm{d}\eta\\ =&\frac{1}{2}\int \frac{\eta}{\eta^2+1}\,\mathrm{d}\eta+\frac{1}{2}\int \frac{1}{\eta^2+1}\,\mathrm{d}\eta\\ =&\frac{1}{4}\ln\left(\eta^2+1\right)+\frac{1}{2}\arctan\eta\\ =&\frac{1}{4}\ln\left(2\left[x^2+\sqrt{2}x+1\right]\right)+\frac{1}{2}\arctan\left(\sqrt{2}x+1\right)+C_1.\end{aligned} \tag{18}$$

A similar calculation gives

$$\int \frac{x-\sqrt{2}}{\left(\sqrt{2}x-1\right)^2+1}\,\mathrm{d}x = \frac{1}{4}\ln\left(2\left[x^2-\sqrt{2}x+1\right]\right)-\frac{1}{2}\arctan\left(\sqrt{2}x-1\right)+C_2. \tag{19}$$

Therefore, with the two constants of integration combined into a single constant,

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^4+1} =&\frac{1}{2\sqrt{2}}\left[\ln\sqrt{\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}}\right.\\ &\left.+\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]+C.\end{aligned} \tag{20}$$

Finally,

$$\begin{aligned}I =&x-\frac{1}{2\sqrt{2}}\left[\ln\sqrt{\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}}\right.\\ &\left.+\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]+C.\end{aligned} \tag{21}$$