# The Summer 2025 Featured Problem Series Week 10: Junior/Senior-Level Ordinary Differential Equations

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## The Archive

This is the last problem from the summer series.

To see problems and solutions in the fall series, which runs from October 13 through December 15 visit The Fall 2025 Featured Problem Series

## Problem

Our final problem for this summer's Featured Problem of the Week series comes from Penn State Math 411, an upper level ordinary differential equations class. The methods used in the solution are taught in a sophomore-level course in differential equations, such as Penn State Math 250 or 251, but applying the tools to assemble a solution requires more mathematical sophistication than can be expected of a sophomore.

This week's problem was selected to illustrate the rich behavior of nonlinear differential equations.

Solve the second-order nonlinear differential equation

$$x''(t) = x'(t)x(t), \tag{1}$$

with initial conditions $x(0) = x_0$ and $x'(0) = x'_0$, by consideration of three cases for the initial conditions

a. $x'_0 = \frac{x_0^2}{2}$,

b. $x'_0 < \frac{x_0^2}{2}$, and

c. $x'_0 > \frac{x_0^2}{2}$.

# Solution

The first step in the solution of $(1)$ is to rewrite it as

$$\left[x'(t) - \frac{1}{2}(x(t))^2\right]' = 0. \tag{2}$$

It follows that

$$x'(t) = \frac{1}{2}(x(t))^2 + C, \tag{3}$$

where $C$ is a constant determined by the initial conditions. This first-order differential equation satisfies the conditions of the Existence and Uniqueness Theorem for first-order differential equations, as discussed at the end of this write-up, thus it is guaranteed to have a unique local solutions satisfying the initial condition $x(0) = x_0$.

Equation $(3)$ is a separable first order equation that is equivalent to

$$\int \frac{\mathrm{d}x}{\frac{1}{2}x^2 + C} = \int \mathrm{d}t = t + K, \tag{4}$$

where $K$ is an arbitrary constant, provided that $\frac{x_0^2}{2} + C \neq 0$. To carry out the integration on the left three case, which correspond to the case in the statement of the problem, are considered:

$\text{(a) } x_0' = \frac{x_0^2}{2}, \text{ (b) } x_0' < \frac{x_0^2}{2}, \text{ and (c) } x_0' > \frac{x_0^2}{2}.$

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$\textbf{Case a: } x_0' = \frac{x_0^2}{2}$

In this case, $C = x_0' - \frac{x_0^2}{2} = 0$. Hence $x = 0$ is an equilibrium of $(3)$. Consequently if $x_0 = 0$, then the solution is $x(t) = 0$ for $t \in \mathbb{R}$.

If $x_0 \neq 0$, then $(4)$ becomes

$$2\int x^{-2}\,\mathrm{d}x = -2x^{-1} = t + K. \tag{5}$$

This yields the solution

$$x(t) = -\frac{2}{t + K}. \tag{6}$$

The initial condition is used to find $K$. The result is

$$x(t) = \frac{2x_0}{2 - x_0 t}. \tag{7}$$

Observe that if $x_0 = 0$, then $(7)$ gives $x(t) = 0$ for all $t$, hence $(7)$ gives the solution for all initial values of $x$ for this case.

The right hand side of $(7)$ is undefined at $t = \frac{2}{x_0}$, if $x_0 \neq 0$. It follows that

$$x(t) = \frac{2x_0}{2 - x_0 t} \quad \text{for} \quad \begin{cases} t \in (-\infty, \infty) & \text{if } x_0 = 0, \\ t \in (-\infty, \tau_\infty) & \text{if } x_0 > 0, \\ t \in (-\tau_\infty, \infty) & \text{if } x_0 < 0, \end{cases} \tag{8}$$

where $\tau_\infty := 2\,|x_0|^{-1}$. Since $(3)$ is equivalent to $(1)$, and the function in $(8)$ is twice differentiable, this is the solution of the second-order nonlinear differential equation $(1)$ whenever $2x_0' = x_0^2$.

The solution has the following asymptotic behavior

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for } x_0 < 0, \text{and} \tag{9}$$

$$\lim_{t\to\tau_\infty^-} x(t) = \infty \qquad \text{for } x_0 > 0. \tag{10}$$

In summary, if the system starts in equilibrium it stays in equilibrium for all $t$. If the system starts to the left of equilibrium ($x_0 < 0$) it will converge to equilibrium as $t \to \infty$. Finally, if the system starts to the right of equilibrium ($x_0 > 0$) it will diverge to infinity in a finite amount of time. Thus the equilibrium is semi-stable.

This classification of the equilibrium can be deduced from a qualitative analysis of $(3)$. However, a qualitative analysis would not lead to the conclusion that the solution blows up in a finite amount of time whenever the system starts to the right of equilibrium.

**Case b:** $x_0' < \frac{x_0^2}{2}$

In this case, $C = x_0' - \frac{x_0^2}{2} < 0$. Set

$$a := \sqrt{-2C} = \sqrt{x_0^2 - 2x_0'}. \tag{11}$$

The equilibrium points of $(3)$ are $x = \pm a$. Thus if $x_0 = \pm a$, then $x(t) = \pm a$ for all $t$ is the solution to the initial value problem.

Suppose that $x_0 \neq \pm a$. Then $(4)$ becomes

$$2\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \mathrm{d}x, \tag{12}$$

where the equality is a consequence of a straightforward partial fraction decomposition. The integral on the right is evaluated term by term, and after simplification, gives

$$\frac{1}{a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \mathrm{d}x = \frac{1}{a}\ln\left|\frac{x-a}{x+a}\right| = t + K. \tag{13}$$

This is equivalent to

$$\left|\frac{x(t)-a}{x(t)+a}\right| = Ae^{at}, \tag{14}$$

where $A = e^{aK}$.

To solve $(14)$ for $x(t)$, it is shown that the absolute value on the left can be dropped. First, note, as shown at the end of this write-up, there exists a closed interval $I$ centered at $0$, such that $x(t)$ satisfies the initial value problem $(3)$ on $I$. Thus $x(t)$ is a differentiable function on the interval, and consequently continuous there. Hence the functions $x(t) - a$ and $x(t) + a$ are both continuous on $I$. Further, the right hand side of $(14)$ is finite for $t \in \mathbb{R}$. Therefore $x(t) + a \neq 0$ for $t \in I$. By the quotient rule for continuous functions

$$g(t) := \frac{x(t) - a}{x(t) + a}$$

is continuous on $I$. In addition, the right hand side of $(3)$ is nonzero for $t \in \mathbb{R}$. Thus $g(t) \neq 0$ for $t \in I$. If a continuous function does not vanish on an interval, then it does not change signs on the interval. Hence the sign of

$$g(0) = \frac{x_0 - a}{x_0 + a} \tag{15}$$

determines the sign of $g$ on $I$. Consequently

$$\left|\frac{x(t) - a}{x(t) + a}\right| = \begin{cases} \dfrac{x(t) - a}{x(t) + a} & \text{for } |x_0| > a, \\ \\ -\dfrac{x(t) - a}{x(t) + a} & \text{for } |x_0| < a. \end{cases} \tag{16}$$

Equation $(14)$ is evaluated at $t = 0$, to find $A$ in terms of the initial conditions.

$$A = \begin{cases} \dfrac{x_0 - a}{x_0 + a} & \text{for } |x_0| > a, \\ \\ -\dfrac{x_0 - a}{x_0 + a} & \text{for } |x_0| < a. \end{cases} \tag{17}$$

With the help of $(16)$ and $(17)$, $(14)$ becomes

$$\frac{x(t) - a}{x(t) + a} = \left[\frac{x_0 - a}{x_0 + a}\right] e^{at}. \tag{18}$$

This is solved for $x(t)$ in terms of the hyperbolic tangent.

$$\begin{aligned} x(t) =& a\frac{1 + \left[\frac{x_0 - a}{x_0 + a}\right] e^{at}}{1 - \left[\frac{x_0 - a}{x_0 + a}\right] e^{at}} \\ =& a\frac{x_0 \cosh \frac{at}{2} - a \sinh \frac{at}{2}}{a \cosh \frac{at}{2} - x_0 \sinh \frac{at}{2}} \\ =& x_0 \frac{1 - \frac{a}{x_0} \tanh \frac{at}{2}}{1 - \frac{x_0}{a} \tanh \frac{at}{2}}. \end{aligned} \tag{19}$$

At this point an observation is in order. The right hand side of $(19)$ is defined everywhere except for $t$ satisfying

$$\tanh\frac{at}{2}=\frac{a}{x_0}. \tag{20}$$

The range of the hyperbolic tangent is $(-1,1)$. Therefore if $|x_0|\leq a$, $x(t)$ is defined on $(-\infty,\infty)$ . In particular, if $x_0=\pm a$, then $(19)$ gives that $x(t)=\pm a$ for $t\in R$. Hence $(19)$ gives the solution for all possible initial values. On the other hand, if $|x_0|>a$, then $x(t)$ is undefined at

$$t(x_0):=\frac{2}{a}\tanh^{-1}\left(\frac{a}{x_0}\right)=\frac{1}{a}\ln\left(\frac{x_0+a}{x_0-a}\right). \tag{21}$$

Note that $t(x_0)>0$ if $x_0>0$ and that $t(x_0)=-t(-x_0)$. Set $\tau:=t(|x_0|)$. The solution to the initial value problem in this case is

$$x(t)=x_0\frac{1-\frac{a}{x_0}\tanh\frac{at}{2}}{1-\frac{x_0}{a}\tanh\frac{at}{2}}\quad\text{for}\quad\begin{cases}t\in\mathbb{R} & \text{if }\ |x_0|\leq a,\\ t\in(-\infty,\tau) & \text{if } x_0>a,\\ t\in(-\tau,\infty) & \text{if} x_0<-a.\end{cases} \tag{22}$$

It follows from the equivalence of $(3)$ and $(1)$, and that $(22)$ is a twice differentiable function, that $(22)$ is the solution of the second-order nonlinear differential equation $(1)$ whenever $x_0'<\frac{x_0^2}{2}$.

The asymptotic behavior of the solution is now examined.For $x_0>a$,

$$\lim_{t\to\tau^-}x(t)=\infty \tag{23}$$

and for $|x|_0<a$ or $x_0<-a$

$$\lim_{t\to\infty}x(t)=-a. \tag{24}$$

Thus $x=a=\sqrt{x_0^2-2x_0'}$ is an unstable equilibrium and $x=-a=-\sqrt{x_0^2-2x_0'}$ is a stable equilibrium. As in Case a, this much could have been deduced from a qualitative analysis of $(22)$. However, the fact that the solution blows up in finite amount of time if the system starts to the right of $a$ would not have been revealed by a qualitative analysis.

**Case c:** $x_0'>\frac{x_0^2}{2}$

In this case, $C=x_0'-\frac{x_0^2}{2}>0$. Set

$$b:=\sqrt{2C}=\sqrt{2x_0'-x_0^2}. \tag{25}$$

There are no equilibrium solutions of $(3)$ in this case. Consequently the solution is given by

$$\frac{2}{b^2}\int\frac{\mathrm{d}x}{1+\left(\frac{x}{b}\right)^2}=\frac{2}{b}\arctan\left(\frac{x}{b}\right)=t+K, \tag{26}$$

for all $x_0$. Hence

$$x(t) = b\tan\left(\frac{bt}{2} + \phi_0\right), \tag{27}$$

where $\phi_0 = \arctan\left(\frac{x_0}{b}\right)$. The range of the arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore the domain of $x(t)$ is

$$J := \left(\frac{2}{b}\left(-\frac{\pi}{2} - \phi_0\right), \frac{2}{b}\left(\frac{\pi}{2} - \phi_0\right)\right). \tag{28}$$

and $|\phi_0| < \frac{\pi}{2}$. From the bounds on $\phi_0$, one can conclude that

$$\tau_1 := \frac{2}{b}\left(-\frac{\pi}{2} - \phi_0\right) < 0 \qquad \text{and} \qquad \tau_2 := \frac{2}{b}\left(\frac{\pi}{2} - \phi_0\right) > 0,$$

therefore $0 \in J$ as it must. The solution to the initial value problem $(3)$ and consequently $(1)$, since it is a twice differentiable function, for any $x_0$ satisfying $2x_0' > x_0^2$ is

$$x(t) = b\tan\left(\frac{bt}{2} + \phi_0\right) \quad t \in (\tau_1, \tau_2). \tag{29}$$

The asymptotics of the solution in this case are straight forward. For all initial conditions

$$\lim_{t \to \tau_2^-} x(t) = \infty, \tag{30}$$

that is the solution blows up in a finite amount of time for all $x_0$. A qualitative analysis of $(3)$ would have led to the conclusion that the solution diverges regardless of the initial value, but not that the solution would do so in finite time for all initial values.

This concludes the discussion of the solution of the initial value problem $(1)$ .

A brief discussion of the Existence and Uniqueness Theorem as it applies to $(3)$ follows.

Showing that the initial value problem

$$x'(t) = \frac{1}{2}\left(x(t)\right)^2 + C \quad x(0) = x_0, \tag{31}$$

where C is a constant, satisfies the conditions of the Existence and Uniqueness Theorem for first-order differential equations may appear redundant, at least with regard to the existence of solutions, since solutions have been exhibited in all cases, which proves solutions exist. However, the existence of a differentiable solution to the initial value problem $(3)$ was used in the course of the solution in Case b. Hence the argument in that case is not complete without this step. Further, showing uniqueness proves that no solutions have been overlooked.

First, recall the theorem.

**Theorem 1.** *(Existence and Uniqueness) Suppose that $D \subseteq \mathbb{R}\times\mathbb{R}$ is a closed rectangle, and $f|D \to R$ is continuous in its first argument, and Lipshitz continuous in its second argument with Lipshitz constant that does not depend on the first argument. Then for every $(t_0, x_0) \in \mathrm{Int}D$, there exists a closed interval $I$ centered on $t_0$ and a $x|I \to \mathbb{R}$ that satisfies the initial value problem*

$$x'(t) = f(t, x(t)), \tag{32}$$

*and $x(t_0) = x_0$.*

In the problem at hand, $f|\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(t, x) = \frac{1}{2}x^2 + C. \tag{33}$$

The right hand side does not depend on $t$. Hence for fixed $x$, $f(t, x)$ is a continuous function of $t$ for $t \in \mathbb{R}$. Next consider $f$ as a function of $x$ for fixed $t$ restricted to a closed and bounded interval $[a, b]$. For $x, y \in \mathbb{R}$.

$$|f(t,x) - f(t,y)| = \left|\frac{x^2 - y^2}{2}\right| = \frac{|x+y|}{2}|x-y| \leq \frac{|x| + |y|}{2}|x-y|, \tag{34}$$

where the inequality follows from the triangle inequality. Set $\kappa := \max\{|a|, |b|\}$. Then for $x, y \in [a, b]$,

$$|f(t,x) - f(t,y)| = \left|\frac{x^2 - y^2}{2}\right| \leq \kappa|x-y|. \tag{35}$$

Thus $f$ is Lipshitz continuous in $x$ on any closed and bound interval $[a, b]$ with Lipshitz constant $\kappa$ which does not depend on $t$. It follows that for $f$ restricted to

$$D = \mathbb{R} \times [a, b] \tag{36}$$

that the initial value problem satisfies Theorem 1. ■