The Summer 2025 Featured Problem Series Week 8: Sophomore-Level Real Analysis

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The Archive

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Problem

We turn to Penn State Math 312, a sophomore-level real analysis course, for this week's problem. This course, along with Math 311W, which was the source of last weeks problem, give math majors their first taste of proof based mathematics. One could summarize the the mission of the course as dotting the "i's" and crossing the "t's" of calculus.

The solution of the problem we have selected for this week will require a fair amount of creative thinking just to get started.

Prove , without the aid of L'Hôpital's rule, or a Taylor's series, that

$$\lim_{x\to 0}\frac{x-\arctan x}{x^3}=\frac{1}{3}. \tag{1}$$

Solution

The key to this proof is representing x and $\arctan x$ as definite integrals as follows

$$x=\int_0^x \mathrm{d}t \qquad \text{and} \qquad \arctan x=\int_0^x \frac{1}{1+t^2}, \mathrm{d}t. \tag{2}$$

With these representation, one has

$$\lim_{x\to 0}\frac{x-\arctan x}{x^3}=\lim_{x\to 0}\frac{1}{x^3}\int_0^x\left[1-\frac{1}{1+t^2}\right]\mathrm{d}t=\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t^2}{1+t^2}\,\mathrm{d}t. \tag{3}$$

Before an attempt is made to evaluate the limit, the u-substitution $u = \frac{t}{x}$ is made in the integral on the right in (3). This gives

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^2}\,\mathrm{d}t = \lim_{x \to 0} \int_0^x \frac{\left(\frac{t}{x}\right)^2}{1+t^2}\,\frac{\mathrm{d}t}{x} = \lim_{x \to 0} \int_0^1 \frac{u^2}{1+x^2u^2}\,\mathrm{d}u. \tag{4}$$

Observe that

$$\lim_{x \to 0} \frac{u^2}{1+x^2u^2} = u^2, \tag{5}$$

and

$$\int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}. \tag{6}$$

Thus the proof can be completed by showing the limit and integral on the right of (4) can be interchanged. To this end, the squeeze theorem is used to show that

$$\lim_{x \to 0} \left[\int_0^1 u^2\,\mathrm{d}u - \int_0^1 \frac{u^2}{1+x^2u^2}\,\mathrm{d}u, \right] = 0. \tag{7}$$

Note that

$$\left| \int_0^1 u^2\,\mathrm{d}u - \int_0^1 \frac{u^2}{1+x^2u^2}\,\mathrm{d}u \right| = \left| \int_0^1 \frac{x^2u^4}{1+x^2u^2}\,\mathrm{d}u \right|, \tag{8}$$

and

$$0 \leq \frac{x^2u^4}{1+x^2u^2} \leq x^2u^4. \tag{9}$$

Hence the monoticity of the integral yields

$$0 \leq \int_0^1 \frac{x^2u^4}{1+x^2u^2}\,\mathrm{d}u \leq x^2 \int_0^1 \,\mathrm{d}u^4 = x^2\frac{1}{5}. \tag{10}$$

Therefore

$$0 \leq \lim_{x \to 0} \int_0^1 \frac{x^2u^4}{1+x^2u^2}\,\mathrm{d}u \leq \frac{1}{5} \lim_{x \to 0} x^2 = 0. \tag{11}$$

Consequently (7) holds by the squeeze theorem. ■