The Summer 2025 Featured Problem Series Week 7: Sophomore-Level Discrete Math

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Problem

This week's problem comes from Penn State Math 311W, a sophomore-level discrete math course known for its concrete yet challenging problems that do not depend on knowledge of calculus.

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined recursively as follows:
$F_0 = 0, F_1 = 1$, and for $n > 1$

$$F_n = F_{n-1} + F_{n-2}. \tag{1}$$

Show that every two successive terms of the Fibonacci sequence are relatively prime, that is $\gcd(F_n, F_{n+1}) = 1$ for $n \geq 0$.

Solution

The proof uses induction.

The base case is $n = 0$. In this case, the definition of the Fibonacci sequence yields

$$\gcd(F_0, F_1) = \gcd(0, 1). \tag{2}$$

Let d be a common divisor of 0 and 1. Although $d|0$ for all d, $d|1$ implies that $d = 1$. Hence $d = 1$ is the only common divisor of F_0 and F_1. In particular, there is no common divisor greater than 1, consequently $\gcd(F_0, F_1) = 1$. Thus the base case holds.

The induction hypothesis is that $\gcd(F_n, F_{n+1}) = 1$ for some $n \geq 0$. It must be shown that the induction hypothesis implies that $\gcd(F_{n+1}, F_{n+2}) = 1$. The definition of the the Fibonacci sequence is used to write

$$\gcd(F_{n+1}, F_{n+2}) = \gcd(F_{n+1}, F_{n+1} + F_n) \tag{3}$$

Two properties of the greatest common divisor, $\gcd(a, b) = \gcd(a, b - a)$ and $\gcd(a, b) = \gcd(b, a)$, are used on the right hands side of (3) to obtain

$$\begin{aligned} \gcd(F_{n+1}, F_{n+1} + F_n) &= \gcd(F_{n+1}, [F_{n+1} + F_n] - F_{n+1}) \\ &= \gcd(F_{n+1}, F_n) = \gcd(F_n, F_{n+1}) = 1, \end{aligned} \tag{4}$$

where the last equality follows from the induction hypothesis. Equations $(3)$and (4) together give

$$\gcd(F_{n+1}, F_{n+2}) = 1. \quad \blacksquare \tag{5}$$