The Summer 2025 Featured Problem Series Week 6: Junior/Senior-Level Real Analysis

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The Archive

To see problems and solutions in the fall series, which runs from October 13 through December 15 visit The Fall 2025 Featured Problem Series

Problem

Two weeks ago our problem came from Penn State Math 403, the upper undergraduate real analysis course. This week we return to that course for our problem. In fact, the problem for this week is a variation of the problem from two weeks ago. And, like the problem from two weeks ago, the solution of this week's problem does not require any advanced theorems developed in Math 403. A student who has completed Calculus I will have most of the tools required to solve the problem this week. However, putting those tools to work will require a level of mathematical sophistication that cannot be expected of a Calculus I student.

Recall that in the Week 4 problem, a bijection f between $[0,1]$ and $(0,1)$ was exhibited which proved that these sets have the same cardinality. The function was defined piecewise on the sets $\{0,1\}$,

$$A_2 = \left\{ \frac{1}{k+1} \middle| k \in \mathbb{N}, k \geq 2 \right\}, \qquad B_2 = \left\{ 1 - \frac{1}{k+1} \middle| k \in \mathbb{N}, k \geq 2 \right\},$$

and $C = (0,1)/(A_2 \cup B_2)$, and took the explicit form

$$f(x) := \begin{cases} \frac{1+x}{3} & \text{for } x = \{0,1\}; \\ \frac{x}{x+1} & \text{for } x \in A_2; \\ \frac{1}{2-x} & \text{for } x \in B_2; \\ x & \text{for } x \in C. \end{cases} \tag{1}$$

The motivation for this construction can be found in the solution of the Week 4 problem, which is in the Problem Archive linked to at the bottom of this page.

a. Use either the $\epsilon - \delta$ or sequential definition of continuity to show that f is not continuous at $x = 0$ and $x = 1$.

b. Prove that a continuous bijection from $[0, 1]$ to $(0, 1)$ does not exist. Hint: Use contradiction and the Intermediate Value Theorem (IVT).

Solution

First, part (a) is examined. Both of the suggested methods will be used. To begin, the sequential definition of continuity will be used to prove that f, as defined in (1), is discontinuous at $x = 1$, and then the $\epsilon - \delta$ definition of continuity will be used to prove that it is discontinuous at $x = 0$.

Recall the sequential definition of continuity for real-valued function of a real variable. A function g is continuous at a point a in its domain if and only if for every sequence $\{a_n\}_{n=1}^{\infty}$ that takes values in the domain of g and satisfies $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} g(a_n) = g(a)$.

To prove discontinuity at a it suffices to exhibit a sequence $\{a_n\}_{n=1}^{\infty}$ that takes values in the domain of g and satisfies $\lim_{n\to\infty} a_n = a$, for which $\lim_{n\to\infty} g(a_n) \neq g(a)$.

Returning to the question of the continuity of f at $x = 1$, for $n \in \mathbb{N}$, set

$$a_n = 1 - \frac{1}{n+2}, \tag{2}$$

and observe that that

$$a_n \in B_2 = \left\{ 1 - \frac{1}{k+1} \middle| k \in \mathbb{N}, k \geq 2 \right\}. \tag{3}$$

Consequently the sequence $\{a_n\}_{n=1}^{\infty}$ is in the domain of f. Further,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2} \right) = 1. \tag{4}$$

On the other hand, for $x \in B_2$, the definition of f gives $f(x) = \frac{1}{2-x}$. Therefore

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{1}{2 - a_n} = 1 \neq \frac{2}{3} = f(1). \tag{5}$$

Thus f is discontinuous at $x = 1$.

The $\epsilon - \delta$ definition of continuity for real-valued function of a real variable, which will be used to prove the discontinuity of f at $x = 0$, is as follows. A function g is continuous at a point a in its domain, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all x in the domain of g satisfying $0 < |x - a| < \delta$

$$|g(x) - g(a)| < \epsilon. \tag{6}$$

To prove discontinuity at a it will suffice to exhibit an $\epsilon > 0$ such for all $\delta > 0$ there exists an x in the domain of g satisfying $0 < |x - a| < \delta$ for which

$$|g(x) - g(a)| \geq \epsilon. \tag{7}$$

An exploratory analysis is undertaken in order to prove the discontinuity of f at $x = 0$. The first issue is a suitable choice for $\epsilon > 0$ such that for all $\delta > 0$ there exists an x in the domain of f satisfying

$$0 < |x| < \delta \qquad \text{and} \qquad |f(x) - f(0)| \geq \epsilon. \tag{8}$$

At this point, explicit forms for $f(0)$ and $f(x)$ are substituted into the inequality on the right in (8). From equation (1), one has $f(0) = \frac{1}{3}$. However, the explicit form of $f(x)$ depends on whether $x \in A_2, x \in B_2$, or $x \in C$. From the inequality on the left in (8), the selected set must have $x = 0$ as a limit point. As can be seen from (3), that every sequence in B_2 converge to 1, thus $x = 0$ is not a limit point of this set, which eliminates B_2 from consideration. Next, note that any sequence in

$$A_2 = \left\{ \frac{1}{k+1} \middle| n \in \mathbb{N}, n \geq 2 \right\} \tag{9}$$

converges to zero. Thus zero is a limit point of this set, and it cannot be eliminated from consideration. Finally, for any $0 < r < 1$,

$$(0, r) \cap C = (0, r) / (A_2 \cup B_2) \neq \emptyset,$$

since $A_2 \cup B_2$ contains only a countable number of points. Thus zero is also a limit point of C. Therefore one may select x in either A_2 or C. Because the function f takes a simple form on C,, that is the choice made here. For $x \in C$, (1) gives $f(x) = x$, and the right hand inequality in (8) becomes

$$\left| x - \frac{1}{3} \right| \geq \epsilon. \tag{10}$$

This is equivalent to

$$x \geq \epsilon + \frac{1}{3} \qquad \text{or} \qquad x \leq -\epsilon + \frac{1}{3}. \tag{11}$$

Observe that $0 \notin C = (0, 1) / (A_2 \cup B_2)$, thus $0 < x = |x|$. On the other hand, if

$$\epsilon < \frac{1}{3} \tag{12}$$

the inequality on the right in (11) gives a meaningful upper bound on $|x|$. One is free to chose any $\epsilon > 0$ which satisfies (12). The choice $\epsilon = \frac{1}{12}$ will be used because it results in a sstraightforward calculation, but in this regard it is not unique.

It follows that for any $x \in C$ satisfying $0 < |x| < \min\{\frac{1}{4}, \delta\} < \delta$, that

$$\left| f(x) - \frac{1}{3} \right| \geq \frac{1}{12} \tag{13}$$

Therefore f is discontinuous at $x = 0$, as claimed.

Finally, part (b) is examined, that is it is proven that a continuous bijection from $[0,1]$ to $(0,1)$ does not exist. The proof goes by contradiction. The first step is prove that if $g|[0,1] \to (0,1)$ is a continuous surjection , then it is not injective. By the definition of an injection, it will suffice to show that for some $c_1, c_2 \in [0,1]$ with $c_1 \neq c_2$, $g(c_1) = g(c_2)$. This will be achieved with the help of the Intermediate Value Theorem (IVT):

Theorem 1 (IVT)**.** *If $h|[a,b]] \to \mathbb{R}$ is continuous and $h(a) \neq h(b)$, then for any L satisfying*

$$\min\{h(a), h(b)\} < L < \max\{h(a), h(b)\}$$

there exists a $c \in (a,b)$ such that $h(c) = L$.

Two case are considered. The first case is $g(0) = g(1)$. In this case, obviously g is not injective.

The second case is $g(0) \neq g(1)$. There are two subcases to this case. The first is that $g(0) < g(1)$ and the other is $g(0) > g(1)$. The proof of the second subcase, which is nearly identical to the proof of the first, is left as an exercise for the reader.

Suppose that $g(0) < g(1)$. Since $g([0,1]) = (0,1)$, one has

$$0 < g(0) < g(1) < 1. \tag{14}$$

Take $m \in (0,1)$ satisfying $0 < m < g(0)$. By assumption, g is a surjection, thus there exist $a \in [0,1]$ such that

$$g(a) = m < g(0) < g(1). \tag{15}$$

The above inequalities give $f(0) \neq f(a) \neq f(1)$. Hence, since f is a well-defined function, $0 \neq a \neq 1$. Consequently $a \in (0,1)$. Consider the interval $[a,1] \subset [0,1]$. The continuity of g on this interval follows from its continuity on the containing interval, thus the IVT holds on the closed interval $[a,1]$. Hence by inequality (15) and the IVT there exists a $c \in (a,1)$ such that $g(c) = g(0)$. However, $c \neq 0$, since $0 < a < c$. Therefore g is not injective.

As an aside, one could have begun by selecting an M satisfying $f(1) < M < 1$, and constructed a proof along similar lines. The reader is encourage to do this.

If a continuous bijection $g|[0,1] \to (0,1)$ existed it would be a continuous surjection, which has been shown implies that g is not an injection; which in turn implies that g is not a bijection. This is a contradiction. Thus a continuous bijection from $[0,1]$ to $(0,1)$ does not exist. ■