The Summer 2025 Featured Problem Series Week 4: Junior/Senior-Level Real Analysis

iMathTutor® • 13 Oct 2025

The Archive

To see problems and solutions in the fall series, which runs from October 13 through December 15 visit The Fall 2025 Featured Problem Series

Problem

Our next problem comes from Penn State Math 403, the upper-level undergraduate real analysis course. We have selected a problem which does not require advanced theorems developed in the course. Yet, its solution is challenging, as it requires a carefully constructed bijection.

Prove that $(0,1)$ and $[0,1]$ have the same cardinality.

Solution

By definition. two sets A and B have the same cardinality if there exists a bijection $f|A \to B$. Thus it will suffice to find a bijection from $[0,1]$ to $(0,1)$.

To get a handle on how to deal with the endpoints when constructing the bijection, a related problem is considered first.

Show that the sets $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_2 := \{2, 3 \ldots\}$ have the same cardinality. Define $g|\mathbb{N} \to \mathbb{N}_2$ by $g(n) = n + 1$. It is straight forward to verify that g has inverse $g^{-1}(n) = n - 1$. Thus, since a function is invertbile if and only if it a bijection, g is a bijection. More generally if $A = \{a_k | k \in \mathbb{N}\}$ $A_2 = \{a_k | k \in \mathbb{N}_2\}$, then $g|A \to A_2$ defined by $g(a_n) = a_{n+1}$ is a bijection Hence A and A_2 have the same cardinality.

To put this to use in the current problem, consider the sets

$$A = \left\{ a_k \middle| a_1 = 0, a_k = \frac{1}{k+1}, k \geq 2 \right\} = \{0, \frac{1}{3}, \frac{1}{4}, \ldots\} \subset [0,1],$$

and

$$B = \left\{ b_k \middle| b_1 = 1, b_k = 1 - \frac{1}{k+1}, k \geq 2 \right\} = \{1, \frac{2}{3}, \frac{3}{4}, \ldots\} \subset [0,1] \,;$$

and the corresponding sets

$$A_2 = \left\{ a_k \middle| a_k = \frac{1}{k+1}, k \geq 2 \right\} = \{\frac{1}{3}, \frac{1}{4}, \ldots\} \subset (0,1) \subset [0,1] \,,$$

and

$$B_2 = \left\{ b_k \middle| b_k = 1 - \frac{1}{k+1}, k \geq 2 \right\} = \{\frac{2}{3}, \frac{3}{4}, \ldots\} \subset (0,1) \subset [0,1] \,.$$

The sets A, B, and $C := [0,1] \,/ (A \cup B)$ form a partition of $[0,1]$; and the sets A_2, B_2, and $C = [0,1] \,/ (A \cup B) = (0,1) \,/ (A_2 \cup B_2)$ form a partition of $(0,1)$. This allows for a bijection to be defined piecewise. Define $f_{|A} \middle| A \to A_2$ by $f_{|A}(a_n) = a_{n+1}$, $f_{|B} \middle| B \to B_2$ by $f_{|B}(b_n) = b_{n+1}$, and $f_{|C} \middle| C \to C$ by $f_{|C}(x) = x$. Finally, define $f | [0,1] \to (0,1)$ by

$$f(x) = \begin{cases} f_{|A}(x) & \text{for } x \in A \\ f_{|B}(x) & \text{for } x \in B \\ f_{|C}(x) & \text{for } x \in C. \end{cases} \tag{1}$$

As noted earlier, $f_{|A}$ and $f_{|B}$ are invertible, while $f_{|C}$ is the identity map on C, so it too is invertible. Consequently f is invertible, with

$$f^{-1}(x) = \begin{cases} f_{|A}^{-1}(x) & \text{for } x \in A_2 \\ f_{|B}^{-1}(x) & \text{for } x \in B_2 \\ f_{|C}^{-1}(x) & \text{for } x \in C. \end{cases} \tag{2}$$

Therefore f is a bijection. Hence it has been shown that $[0,1]$ and $(0,1)$ have the same cardinality.

Although the problem has been solved at this point, the it would be nice to have an explicit form for f. To this end, note that

$$f(x) = \begin{cases} \frac{1+x}{3} & \text{for } x \in \{0,1\} \\ \frac{x}{x+1} & \text{for } x \in A_2 \\ \frac{1}{2-x} & \text{for } x \in B_2 \\ x & \text{for } x \in C. \end{cases} \tag{3}$$

■