The Summer 2025 Featured Problem Series Week 4: Junior/ Senior-Level Real Analysis

iMathTutor® • 13 Oct 2025

The Archive

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Problem

Our next problem comes from Penn State Math 403, the upper-level undergraduate real analysis course. We have selected a problem which does not require advanced theorems developed in the course. Yet, its solution is challenging, as it requires a carefully constructed bijection.

Prove that $(0,1)$ and $[0,1]$ have the same cardinality.

Solution

By definition. two sets A and B have the same cardinality if there exists a bijection $f|A \to B$. Thus it will suffice to find a bijection from $[0,1]$ to $(0,1)$.

To get a handle on how to deal with the endpoints when constructing the bijection, a related problem is considered first.

Show that the sets $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_2 := \{2,3\ldots\}$ have the same cardinality. Define $g|\mathbb{N} \to \mathbb{N}_2$ by $g(n) = n+1$. It is straight forward to verify that g has inverse $g^{-1}(n) = n-1$. Thus, since a function is invertbile if and only if it a bijection, g is a bijection. More generally if $A = \{a_k | k \in \mathbb{N}\}$ $A_2 = \{a_k | k \in \mathbb{N}_2\}$, then $g|A \to A_2$ defined by $g(a_n) = a_{n+1}$ is a bijection Hence A and A_2 have the same cardinality.

To put this to use in the current problem, consider the sets

$$A = \left\{ a_k \middle| a_1 = 0, a_k = \frac{1}{k+1}, k \geq 2 \right\} = \{0, \frac{1}{3}, \frac{1}{4}, \ldots\} \subset [0,1]\,,$$

and

$$B = \left\{ b_k \middle| b_1 = 1, b_k = 1 - \frac{1}{k+1}, k \geq 2 \right\} = \{1, \frac{2}{3}, \frac{3}{4}, \ldots\} \subset [0,1]\,;$$

and the corresponding sets

$$A_2 = \left\{ a_k \middle| a_k = \frac{1}{k+1}, k \geq 2 \right\} = \{\frac{1}{3}, \frac{1}{4}, \ldots\} \subset (0,1) \subset [0,1]\,,$$

and

$$B_2 = \left\{ b_k \middle| b_k = 1 - \frac{1}{k+1}, k \geq 2 \right\} = \{\frac{2}{3}, \frac{3}{4}, \ldots\} \subset (0,1) \subset [0,1]\,.$$

The sets A, B, and $C := [0,1]\,/(A \cup B)$ form a partition of $[0,1]$; and the sets A_2, B_2, and $C = [0,1]\,/(A \cup B) = (0,1)\,/(A_2 \cup B_2)$ form a partition of $(0,1)$. This allows for a bijection to be defined piecewise. Define $f_{|A} \big| A \to A_2$ by $f_{|A}(a_n) = a_{n+1}$, $f_{|B} \big| B \to B_2$ by $f_{|B}(b_n) = b_{n+1}$, and $f_{|C} \big| C \to C$ by $f_{|C}(x) = x$. Finally, define $f | [0,1] \to (0,1)$ by

$$f(x) = \begin{cases} f_{|A}(x) & \text{for } x \in A \\ f_{|B}(x) & \text{for } x \in B \\ f_{|C}(x) & \text{for } x \in C. \end{cases} \tag{1}$$

As noted earlier, $f_{|A}$ and $f_{|B}$ are invertible, while $f_{|C}$ is the identity map on C, so it too is invertible. Consequently f is invertible, with

$$f^{-1}(x) = \begin{cases} f^{-1}_{|A}(x) & \text{for } x \in A_2 \\ f^{-1}_{|B}(x) & \text{for } x \in B_2 \\ f^{-1}_{|C}(x) & \text{for } x \in C. \end{cases} \tag{2}$$

Therefore f is a bijection. Hence it has been shown that $[0,1]$ and $(0,1)$ have the same cardinality.

Although the problem has been solved at this point, the it would be nice to have an explicit form for f. To this end, note that

$$f(x) = \begin{cases} \frac{1+x}{3} & \text{for } x \in \{0,1\} \\ \frac{x}{x+1} & \text{for } x \in A_2 \\ \frac{1}{2-x} & \text{for } x \in B_2 \\ x & \text{for } x \in C. \end{cases} \tag{3}$$

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