The Summer 2025 Featured Problem Series Week 2: Junior/Senior-Level Probability

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The Archive

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Problem

This week our problem comes from Penn State Math 414 an upper division probability course that does not use measure theory. Unlike many of the problems in this course, the random variable is not one of the commom named random variables. So the solution will take some thought.

Consider a sequence of independent Bernoulli trials with fixed probability of success p. The trials will end after the first success is observed or after n trials if no success has occurred by then. Let X denote the number of trials conducted. Find $\mathbf{E}\left[X\right]$.

Solution

Consider the related procedure, a sequence of independent Bernoulli trials with fixed probability of success p terminates after the observation of the first success. A well-known result for this procedure is that the number of trials before termination Y is a geometric random variable with parameter p. The solution of the given problem will proceed by expressing X in terms of Y and exploiting known properties of geometric random variables.

To express X in terms of Y a new new Bernoulli random variable Z is introduced. It is defined by

$$Z := \begin{cases} 1 & \text{for } Y \leq n, \\ 0 & \text{for } Y > n, \end{cases} \tag{1}$$

With this new random variable one may express X in terms of Y as follows

$$X = YZ + n(1 - Z). \tag{2}$$

To exploit the fact that the geometric distribution is memoryless, this equation is rearranged to give

$$X = Y - Y(1 - Z) + n(1 - Z) = Y - (Y - n)(1 - Z). \tag{3}$$

By the memoryless property, $Y - n|Y > n \sim \mathrm{G}(p)$ or equivalently $Y - n|Z = 0 \sim \mathrm{G}(p)$.

Now, $\mathbf{E}[X]$ is found. The linearity of expectation and the well-known fact that $\mathbf{E}[Y] = \frac{1}{p}$ yields

$$\mathbf{E}[X] = \mathbf{E}[Y] - \mathbf{E}[Y(1 - Z)] = \frac{1}{p} - \mathbf{E}[(Y - n)(1 - Z)]. \tag{4}$$

Conditioning on the value of Z is used to find $\mathbf{E}[(Y - n)(1 - Z)]$:

$$\begin{aligned}
\mathbf{E}[(Y - n)(1 - Z)] =& \mathbf{E}[(Y - n)(1 - Z)|Z = 0]\, \mathbf{P}[Z = 0] \\
& + \mathbf{E}[(Y - n)(1 - Z)|Z = 1]\, \mathbf{P}[Z = 1] \\
=& \mathbf{E}[(Y - n)(1 - Z)|Z = 0]\, \mathbf{P}[Z = 0] \\
=& \mathbf{E}[Y - n|Z = 0]\, \mathbf{P}[Z = 0] = \frac{1}{p}\mathbf{P}[Z = 0].
\end{aligned} \tag{5}$$

The second and third equalities in the above sequence of equalities follows from $1 - Z|Z = 1 \equiv 0$, and $1 - Z|Z = 0 \equiv 1$, respectively. The fourth equality is a consequence of the memoryless property. Finally, observe that $[Z = 0] = [Y > n]$. This is the event that it takes more than n trials for the sequence of trials to terminate. which will occur if and only if the first n trials are all failure. Thus by independence of the trials and constant probability of success,

$$\mathbf{P}[Z = 0] = (1 - p)^n \tag{6}$$

Equations (4),(5), and (6) are combined to give

$$\mathbf{E}[X] = \frac{1}{p} - \frac{1}{p}(1 - p)^n = \frac{1}{p}[1 - (1 - p)^n]. \quad \blacksquare \tag{7}$$