The Summer 2025 Featured Problem Series Week 1: Calculus I

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The Archive

To see problems and solutions in the fall series, which runs from October 13 through December 15 visit The Fall 2025 Featured Problem Series

Problem

We thought we would ease into the series with a Penn State Math 140, i.e. Calculus I, problem. But don't be fooled, it is a bit more challenging than a typical Calc I problem, since its solution doesn't involve identifying the type of problem and applying an established algorithmic procedure.

Suppose that for some $a < b$, f and g are continuous functions on $[a, b]$, and differentiable on (a, b). Show that if $f(a) = g(a)$ and $f'(x) < g'(x)$ on (a, b), then $f(b) < g(b)$.

Solution

Let $h(x) := g(x) - f(x)$ on $[a, b]$. The difference of two functions is continuous if the functions are continuous, and differentiable if the functions are differentiable. Hence, by the assumptions of the problem, h is continuous on $[a, b]$, and diffentiable on (a, b). Consequently h satisfies the conditions of the Mean Value Theorem (MVT). Therefore there exists a $c \in (a, b)$, such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}. \tag{1}$$

The rule for differentiation of the difference of two differentiable functions, gives $h'(c) = g'(c) - f'(c) > 0$, where the inequality follows from the assumption that $f'(x) < g'(x)$ on (a, b). Hence

$$\frac{h(b) - h(a)}{b - a} > 0 \tag{2}$$

From the definition of h, $h(b) = g(b) - f(b)$, furthermore $h(a) = g(a) - f(a) = 0$, since it is assumed that $f(a) = g(a)$. Observe that $a < b$ implies $b - a > 0$, thus the inequality (2) yields

$$g(b) - f(b) > 0, \tag{3}$$

which shows that $f(b) < g(b)$. ■