Electromagnetic puzzle

110 • 13 Sep 2025

For years I have been thinking about situations where the momentum seemed to not be conserved in electromagnetism. Let's discuss one here, I am still not at ease with the conclusion and it is sadly not a settled problem in my head. I have no doubt about the conservation laws but their interpretation in certain situations can be challenging.

In classical electromagnetism, radiation pressure arises from the transfer of momentum carried by electromagnetic waves. We can quantify this using the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ and the energy density $u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$. In particular, the radiation pressure on a perfectly absorbing surface is $P = \frac{\langle S \rangle}{c}$, where $\langle S \rangle$ is the time-averaged Poynting flux. Likewise, in the restframe of the reflecting surface, for a perfectly reflecting surface and considering the acceleration of the reflector is negligible, the pressure is twice as big as for the absorbing surface: $P = \frac{2\langle S \rangle}{c}$.

Linking this to the quantum picture, radiation can also be described as a collection of photons, each carrying momentum $p = \frac{h\nu}{c}$. Since photons carry momentum, their absorption or reflection transfer a corresponding momentum to the surface.

More scrutinity on the precise interaction between the reflector and the radiation leads to interpret the radiation pressure as a force on the charge carriers within a conductor, whose origin is nothing but the Lorentz force. Oscillating electric fields from the incoming and reflected radiations drive the electrons, while the associated magnetic fields bend their trajectories, generating a macroscopic force parallel to the normal of the reflector. Keyly, if the electric field did not accelerate the electrons in the plane of the conductor, there would be no net force normal to the surface. Electric forces parallel to the surface are therefore the primary drivers, while the magnetic component redirects their motion to produce the radiation pressure in the wave's propagation direction. Yet, the combined action reproduces the classical expression for radiation pressure on metals.

One can now consider a paradoxical situation. Under the assumption that a charge can be arbitrary massive, a charge passing through a Gaussian beam along the direction of the electric field at its waist experiences a nonzero momentum depending on the initial phase. Let us assume the charge is so massive that its acceleration is negligible in all situations (or we find a way to compensate the acceleration by applying an opposite force of different nature), so the charge does not radiate significantly, and it is travelling at an almost constant speed. In this case, the momentum acquired by the charge cannot be transferred to the radiation and yet the charge acquired a net momentum. Keeping two identical Gaussian beams facing each other, so that the magnetic fields cancel at the waist, isolates the purely electric interaction. Expressing the electric field of two Gaussian beams facing each other as

$$\mathbf{E}_1(\mathbf{r},t) = \frac{E_0}{w(z)} e^{-\frac{y^2+z^2}{w_0^2}} \cos(kx - \omega t + \phi_1)\,\hat{\mathbf{y}}$$

,

$$\mathbf{E}_2(\mathbf{r},t) = \frac{E_0}{w(z)} e^{-\frac{y^2+z^2}{w_0^2}} \cos(-kx - \omega t + \phi_2)\,\hat{\mathbf{y}}.$$

$$\mathbf{E}_{\rm tot}(0,y,t) = 2E_0 e^{-\frac{y^2}{w_0^2}} \cos\big(\omega t - \Phi\big)\,\hat{\mathbf{y}}, \qquad \Phi := \frac{\phi_1+\phi_2}{2},$$

provided the two beams have the same amplitude and waist and we set the relative-phase factor for constructive combination.

$$\Delta p_y \;=\; q\int_{-\infty}^{\infty} E_{{\rm tot},y}\big(0,y(t),t\big)\,dt = 2qE_0\int_{-\infty}^{\infty} e^{-\frac{(vt)^2}{w_0^2}} \cos(\omega t - \Phi)\,dt.$$

$$\text{With } a = \frac{v^2}{w_0^2}, \quad \int_{-\infty}^{\infty} e^{-at^2}\cos(\omega t - \Phi)\,dt = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}\cos\Phi.$$

$$\boxed{\Delta p_y = 2qE_0\sqrt{\pi}\,\frac{w_0}{v}\,e^{-\frac{\omega^2 w_0^2}{4v^2}}\cos\Phi}.$$

$$\Delta p_y^{(\text{phase max})} = 2qE_0\sqrt{\pi}\,\frac{w_0}{v}\,e^{-\frac{\omega^2 w_0^2}{4v^2}}.$$

$$v_{\rm opt} = \frac{\omega w_0}{\sqrt{2}}, \qquad \Delta p_y^{\max} = 2\sqrt{2\pi}\,e^{-1/2}\,\frac{qE_0}{\omega}.$$

$$2\sqrt{2\pi}\,e^{-1/2} \approx 3.0406938021, \qquad \Delta p_y^{\max} \approx 3.0407\,\frac{qE_0}{\omega}.$$

At this stage, it is clear that no radiation carries away this momentum because

$$\mathbf{a} = \frac{\mathbf{F}}{m} \xrightarrow[m\to\infty]{} 0$$

,

the $v_{\rm opt}$ should not be taken seriously as it implies a faster than c speed.

Since the field's momentum density related to the interaction is the result of either

$$\mathbf{E}_{\text{charge}} \times \mathbf{B}_{\text{beam}}$$

or

$$\mathbf{E}_{\text{beam}} \times \mathbf{B}_{\text{charge}}$$

,

the field's momentum density evolves in

$$\mathbf{P}_{\text{field}} \propto \frac{1}{r^3}\hat{\mathbf{y}}$$

and its integral over a spherical shell of arbbitrary thickness evolves like

$$\log\left(\frac{r_{max}}{r_{min}}\right) \xrightarrow[r_{min}\to\infty]{} 0$$

,

thus the field's momentum seems to not be conserved in a moving volume and needs to be transferred somewhere else. Assuming the not unreasonable hypothesis that the momentum is conserved in the system, the first reasonable candidate for the momentum transfer should be the hidden momentums of the sources of the beam, yet, there are arguments that make me question it, the second candidate is to integrate the fields over the whole space, but this should be discarded by thinking of the beam as a finite wave train with a minimal and maximal radius, a third hypothesis would be that the gaussian beams are not realistic beams and no realistic beams could transfer momentum in such a way, while I have some good arguments for that position, they are nullified by the fact that such a charge would also get a non zero force from the wave emitted by an oscillating dipole.

Letting $\mathbf{P}_{\text{hidden}}$ denote the hidden momentum in the source currents, we have

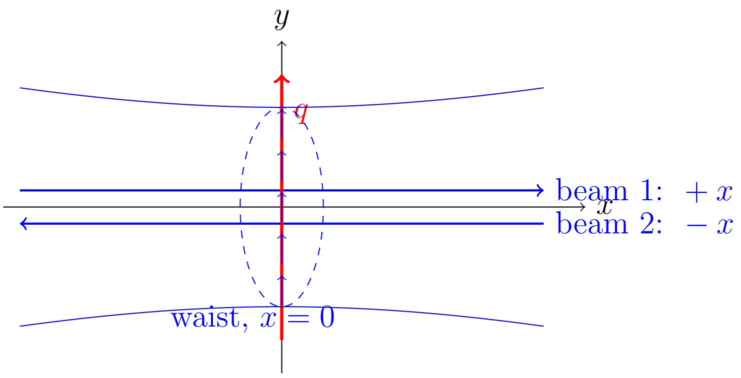
$$\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{charge}} + \mathbf{P}_{\text{fields}} + \mathbf{P}_{\text{hidden}} = 0.$$

Mathematically, the hidden momentum in a source with current density $\mathbf{J}$ and vector potential $\mathbf{A}$ is

$$\mathbf{P}_{\text{hidden}} = \frac{1}{c^2}\int \mathbf{J} \times \mathbf{A}\, d^3r.$$

Obviously, this term should compensate the momentum acquired by the massive charge. No simple exchange with radiated momentum occurs in this idealized setup, highlighting the subtle role of hidden momentums in ensuring total momentum conservation.

Illustration of the Setup



Interestingly, the angular momentum of the field does not seem to vanish like the linear momentum because of its definition

$$\mathbf{L}_{\text{field}} \propto \mathbf{r} \times \mathbf{E} \times \mathbf{B}$$

Which is super weird because it would mean that the angular momentum could be conserved in the fields while the linear momentum would be conserved elsewhere, in hidden momentums. To be honnest, it does not make a lot of sense to me. Now another question is the conservation of the center of energy.

Given the definition of the total energy as

$$U = \int_V u dV$$

where $u = u_{\text{field}} + u_{\text{matter}}$ is the local energy density, and V is the volume of the system, the center of energy $\mathbf{R}_{\text{E}}$ is defined as

$$\mathbf{R}_{\text{E}} U = \int_V \mathbf{r} u dV \Leftrightarrow \mathbf{R}_{\text{E}} = \frac{1}{U} \int_V \mathbf{r} u dV$$