Electromagnetic puzzle

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For years I have been thinking about situations where the momentum seemed to not be conserved in electromagnetism. Let's discuss one here, I am still not at ease with the conclusion and it is sadly not a settled problem in my head. I have no doubt about the conservation laws but their interpretation in certain situations can be challenging.

In classical electromagnetism, radiation pressure arises from the transfer of momentum carried by electromagnetic waves. We can quantify this using the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ and the energy density $u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$. In particular, the radiation pressure on a perfectly absorbing surface is $P = \frac{\langle S \rangle}{c}$, where $\langle S \rangle$ is the time-averaged Poynting flux. Likewise, in the restframe of the reflecting surface, for a perfectly reflecting surface and considering the acceleration of the reflector is negligible, the pressure is twice as big as for the absorbing surface: $P = \frac{2\langle S \rangle}{c}$.

Linking this to the quantum picture, radiation can also be described as a collection of photons, each carrying momentum $p = \frac{h\nu}{c}$. Since photons carry momentum, their absorption or reflection transfer a corresponding momentum to the surface.

More scrutinity on the precise interaction between the reflector and the radiation leads to interpret the radiation pressure as a force on the charge carriers within a conductor, whose origin is nothing but the Lorentz force. Oscillating electric fields from the incoming and reflected radiations drive the electrons, while the associated magnetic fields bend their trajectories, generating a macroscopic force parallel to the normal of the reflector. Keyly, if the electric field did not accelerate the electrons in the plane of the conductor, there would be no net force normal to the surface. Electric forces parallel to the surface are therefore the primary drivers, while the magnetic component redirects their motion to produce the radiation pressure in the wave's propagation direction. Yet, the combined action reproduces the classical expression for radiation pressure on metals.

One can now consider a paradoxical situation. Under the assumption that a charge can be arbitrary massive, a charge passing through a Gaussian beam along the direction of the electric field at its waist experiences a nonzero momentum depending on the initial phase. Let us assume the charge is so massive that its acceleration is negligible in all situations (or we find a way to compensate the acceleration by applying an opposite force of different nature), so the charge does not radiate significantly, and it is travelling at an almost constant speed. In this case, the momentum acquired by the charge cannot be transferred to the radiation and yet the charge acquired a net momentum. Keeping two identical Gaussian beams facing each other, so that the magnetic fields cancel at the waist, isolates the purely electric interaction. Expressing the electric field of two Gaussian beams facing each other as

$$\mathbf{E}_1(\mathbf{r},t) = \frac{E_0}{w(z)} e^{-\frac{y^2+z^2}{w_0^2}} \cos(kx - \omega t + \phi_1)\,\hat{\mathbf{y}}$$

,

$$\mathbf{E}_2(\mathbf{r},t) = \frac{E_0}{w(z)} e^{-\frac{y^2+z^2}{w_0^2}} \cos(-kx - \omega t + \phi_2)\,\hat{\mathbf{y}}.$$

$$\mathbf{E}_{\text{tot}}(0,y,0,t) = 2E_0 e^{-\frac{y^2}{w_0^2}} \cos\big(\omega t - \Phi\big)\,\hat{\mathbf{y}}, \qquad \Phi := \frac{\phi_1+\phi_2}{2},$$

provided the two beams have the same amplitude and waist and we set the relative-phase factor for constructive combination.

$$\Delta p_y \;=\; q\int_{-\infty}^{\infty} E_{\text{tot},y}\big(0,y(t),t\big)\,dt = 2qE_0\int_{-\infty}^{\infty} e^{-\frac{(vt)^2}{w_0^2}} \cos(\omega t - \Phi)\,dt.$$

$$\text{With } a = \frac{v^2}{w_0^2}, \qquad \int_{-\infty}^{\infty} e^{-at^2}\cos(\omega t - \Phi)\,dt = \sqrt{\frac{\pi}{a}}\,e^{-\frac{\omega^2}{4a}}\cos\Phi.$$

$$\boxed{\Delta p_y = 2qE_0\sqrt{\pi}\,\frac{w_0}{v}\,e^{-\frac{\omega^2 w_0^2}{4v^2}}\cos\Phi}.$$

$$\Delta p_y^{(\text{phase max})} = 2qE_0\sqrt{\pi}\,\frac{w_0}{v}\,e^{-\frac{\omega^2 w_0^2}{4v^2}}.$$

$$v_{\text{opt}} = \frac{\omega w_0}{\sqrt{2}}, \qquad \Delta p_y^{\max} = 2\sqrt{2\pi}\,e^{-1/2}\,\frac{qE_0}{\omega}.$$

$$2\sqrt{2\pi}\,e^{-1/2} \approx 3.0406938021, \qquad \Delta p_y^{\max} \approx 3.0407\,\frac{qE_0}{\omega}.$$

At this stage, it is clear that no radiation carries away this momentum because

$$\mathbf{a} = \frac{\mathbf{F}}{m} \xrightarrow[m\to\infty]{} 0$$

,

the v_{opt} should not be taken seriously as it implies a faster than c speed.

Since the field's momentum density related to the interaction is the result of either

$$\mathbf{E}_{\text{charge}} \times \mathbf{B}_{\text{beam}}$$

or

$$\mathbf{E}_{\text{beam}} \times \mathbf{B}_{\text{charge}}$$

,

the field's momentum density evolves in

$$\mathbf{P}_{\text{field}} \propto \frac{1}{r^3}\hat{\mathbf{y}}$$

and its integral over a spherical shell of arbitrary thickness evolves like

$$\log\left(\frac{r_{max}}{r_{min}}\right) \xrightarrow[r_{min}\to\infty]{} 0$$

,

thus the field's momentum seems to not be conserved in a moving volume and needs to be transferred somewhere else. Assuming the not unreasonable hypothesis that the momentum is conserved in the system, the first reasonable candidate for the momentum transfer should be the hidden momentums of the sources of the beam, yet, there are arguments that make me question it, the second candidate is to integrate the fields over the whole space, but this should be discarded by thinking of the beam as a finite wave train with a minimal and maximal radius, a third hypothesis would be that the gaussian beams are not realistic beams and no realistic beams could transfer momentum in such a way, while I have some good arguments for that position, they are nullified by the fact that such a charge would also get a non zero force from the wave emitted by an oscillating dipole.

Letting $\mathbf{P}_{\text{hidden}}$ denote the hidden momentum in the source currents, we have

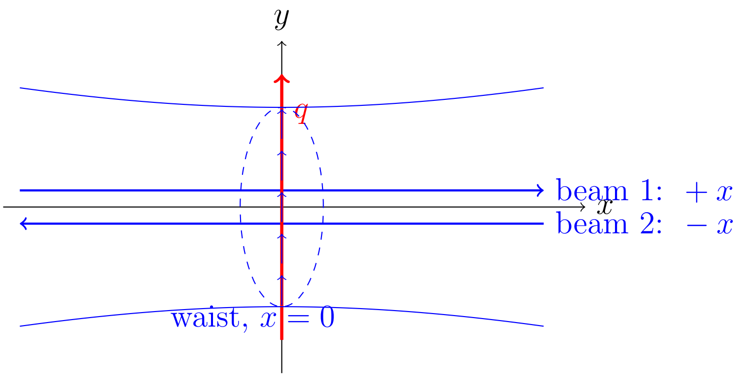
$$\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{charge}} + \mathbf{P}_{\text{fields}} + \mathbf{P}_{\text{hidden}} = 0.$$

Mathematically, the hidden momentum in a source with current density $\mathbf{J}$ and vector potential $\mathbf{A}$ is

$$\mathbf{P}_{\text{hidden}} = \frac{1}{c^2}\int \mathbf{J} \times \mathbf{A}\, d^3r.$$

Obviously, this term should compensate the momentum acquired by the massive charge. No simple exchange with radiated momentum occurs in this idealized setup, highlighting the subtle role of hidden momentums in ensuring total momentum conservation.

Illustration of the Setup



Interestingly, the angular momentum of the field does not seem to vanish like the linear momentum because of its definition

$$\mathbf{L}_{\text{field}} \propto \mathbf{r} \times \mathbf{E} \times \mathbf{B}$$

Which is super weird because it would mean that the angular momentum could be conserved in the fields while the linear momentum would be conserved elsewhere, in hidden momentums. To be honnest, it does not make a lot of sense to me. Now another question is the conservation of the center of energy.

Given the definition of the total energy as

$$U = \int_V u dV$$

where $u = u_{\text{field}} + u_{\text{matter}}$ is the local energy density, and V is the volume of the system, the center of energy $\mathbf{R}_{\text{E}}$ is defined as

$$\mathbf{R}_{\text{E}} U = \int_V \mathbf{r} u dV \Leftrightarrow \mathbf{R}_{\text{E}} = \frac{1}{U} \int_V \mathbf{r} u dV$$

Assuming the conservation of the total energy of the system and an arbbitrary separation between the field energy density and the matter energy density we get

$$
\begin{aligned}
\frac{d\mathbf{R}_{\mathrm{E}}U}{dt} &= \frac{d}{dt}\int_V \mathbf{r}u dV \\
U\frac{d\mathbf{R}_{\mathrm{E}}}{dt} &= \int_V \mathbf{r}\frac{\partial u}{\partial t}dV \\
&= \int_V \mathbf{r}\frac{\partial\left(u_{\text{field}} + u_{\text{matter}}\right)}{\partial t}dV \\
&= \int_V \mathbf{r}\frac{\partial u_{\text{matter}}}{\partial t}dV + \int_V \mathbf{r}\frac{\partial u_{\text{field}}}{\partial t}dV \\
&= \int_V \mathbf{r}\frac{\partial u_{\text{matter}}}{\partial t}dV - \int_V \mathbf{r}\frac{\nabla(\mathbf{E}\times\mathbf{B})}{\mu_0}dV - \int_V \mathbf{r}\left(\mathbf{j}.\mathbf{E}\right)dV \\
&= \int_V \mathbf{r}\frac{\partial u_{\text{matter}}}{\partial t}dV + \int_V \frac{\left(\mathbf{E}\times\mathbf{B}\right)}{\mu_0}dV - \oint_{\partial V} \mathbf{r}\frac{\left(\mathbf{E}\times\mathbf{B}\right).d\mathbf{S}}{\mu_0} - \int_V \mathbf{r}\left(\mathbf{j}.\mathbf{E}\right)dV
\end{aligned}
$$

One term should catch our attention, it is the term

$$\oint_{\partial V} \mathbf{r}\frac{\left(\mathbf{E}\times\mathbf{B}\right).d\mathbf{S}}{\mu_0}$$

For a spherical shell, since no beam can not diverge, at constant radius and fixed θ and ϕ, if $\mathbf{P}_{\text{field}} \propto \frac{1}{r^3}$, then

$$\mathbf{r}\frac{\left(\mathbf{E}\times\mathbf{B}\right).d\mathbf{S}}{\mu_0}$$

is a constant term, (likely fairly tiny), non zero on the edge of the beam and null in the center. But because it comes from the cross fields, it does not really follow the usual picture of energy flow. When $\|\mathbf{E}\|$ is at its maximum, there is a "flow" of energy through the surface coming in the beam at the positive y and going "out" of the beam at negative y, for a radius just half a wavelength bigger, the relation is reversed so it witnesses the circulation of the pseudo energy flow created by the cross fields. That term can be mostly cancelled by an opposite charge following the first at the same speed whose interaction with the beam is just delayed by $\frac{T}{2}$. Thus it can be removed from the relation and we get.

$$\frac{d\mathbf{R}_{\mathrm{E}}U}{dt} = \int_V \mathbf{r}\frac{\partial u_{\text{matter}}}{\partial t}dV + \int_V \frac{\left(\mathbf{E}\times\mathbf{B}\right)}{\mu_0}dV - \int_V \mathbf{r}\left(\mathbf{j}.\mathbf{E}\right)dV$$

In the term $\frac{\partial u_{\text{matter}}}{\partial t}$ we get the variation of kinetic energy from the work of the electric field on the single charge moving and the variation in potential or kinetic energy of the rest of the matter, like sources, through hidden momentums. Since the origin of those hidden momentums are the work of electric forces, the continuity equation gives us

$$\frac{\partial u_{\text{matter}}}{\partial t} = \mathbf{j}.\mathbf{E} - \nabla\mathbf{P}_{\text{matter}}$$

In the end after cancellation of several terms, we get:

$$\frac{d\mathbf{R}_{\mathrm{E}}U}{dt} = \mathbf{P}_{\mathrm{matter}} + \int_V \frac{(\mathbf{E}\times\mathbf{B})}{\mu_0}dV$$

$$\frac{d\mathbf{R}_{\mathrm{E}}}{dt} = \frac{1}{U}\mathbf{P}_{\mathrm{matter}} + \frac{1}{U}\int_V \frac{(\mathbf{E}\times\mathbf{B})}{\mu_0}dV$$

Going back to the expression

$$\frac{\partial u_{\mathrm{field}}}{\partial t} = -\frac{\nabla(\mathbf{E}\times\mathbf{B})}{\mu_0}$$

in the absence of electric current. That expression does not state that the "lump" of energy $u_{\mathrm{field}}dV$ travels in the direction of $\mathbf{E}\times\mathbf{B}$ especially for composite/cross fields (having different sources for the electric and magnetic field). It states that the local variations of energy are equal (up to the sign) to the spatial variation/creation of the Poynting vector. The electric and magnetic fields are divergenceless fields but not the Poynting vector, and the variation of energy density has no obligation to be the result of the displacement of some energy density in the direction of the Poynting vector. When several sources are involved then part of the variation of the field energy density is due to the variation of a scalar product between electric fields. That scalar product can vary because of fields amplitude...or the relative orientations of fields between them. Stated otherwise, the momentum locally created or destroyed is the result of the variations of energy but not necessarily the result of that energy being thrown in that direction. Therefore the vector $\mathbf{r}\frac{\nabla(\mathbf{E}\times\mathbf{B})}{\mu_0}$ witnesses a "virtual" energy displacement result of the cross product of an electric and magnetic field that can be completely unrelated and thus it’s hard, at least fairly locally, to see $\mathbf{R}_E$ as the "real" center of energy/mass, since the $\mathbf{r}$ do not follow any real flow of energy. The energy of a wave can be tracked along the propagation of its field, and the field energy of a charge follows the motion of the charge. Their interference is reflected by the scalar product between their respective fields. Let’s illustrate it with a calculation. Say, a beam propagates mostly along the x direction with a dominating electric component E_y, and a dominating magnetic component B_z. The charge experiencing the interaction is moving in the y direction and has a field

$$\mathbf{E}_{\mathrm{charge}} \approx \frac{q\mathbf{r}-vt\hat{\mathbf{y}}}{4\pi\epsilon_0\|\mathbf{r}-vt\hat{\mathbf{y}}\|^3} = \frac{q\mathbf{n}}{4\pi\epsilon_0\|\mathbf{r}-vt\hat{\mathbf{y}}\|^2}$$

neglecting relativistic effects. Its magnetic field is

$$\propto \mathbf{n}\times\mathbf{E}$$

.

The total energy density is

$$u_{\text{field}} = \frac{\epsilon_0}{2}(\mathbf{E}_{\text{beam}} + \mathbf{E}_{\text{charge}})^2 + \frac{1}{2\mu_0}(\mathbf{B}_{\text{beam}} + \mathbf{B}_{\text{charge}})^2$$

So the developped expression is

$$u_{\text{field}} = \frac{\epsilon_0}{2}(\mathbf{E}^2_{\text{beam}} + 2\mathbf{E}_{\text{beam}}.\mathbf{E}_{\text{charge}} + \mathbf{E}^2_{\text{charge}}) + \frac{1}{2\mu_0}(\mathbf{B}^2_{\text{beam}} + 2\mathbf{B}_{\text{beam}}.\mathbf{B}_{\text{charge}} + \mathbf{B}^2_{\text{cha}}$$

Because the terms $\mathbf{E}^2_{\text{beam}}, \mathbf{B}^2_{\text{beam}}, \mathbf{E}^2_{\text{charge}}, \mathbf{B}^2_{\text{charge}}$ would exist on their own without interactions, the terms that interest us are

$$u_{\text{field}} = \epsilon_0(\mathbf{E}_{\text{beam}}.\mathbf{E}_{\text{charge}}) + \frac{1}{\mu_0}(\mathbf{B}_{\text{beam}}.\mathbf{B}_{\text{charge}})$$

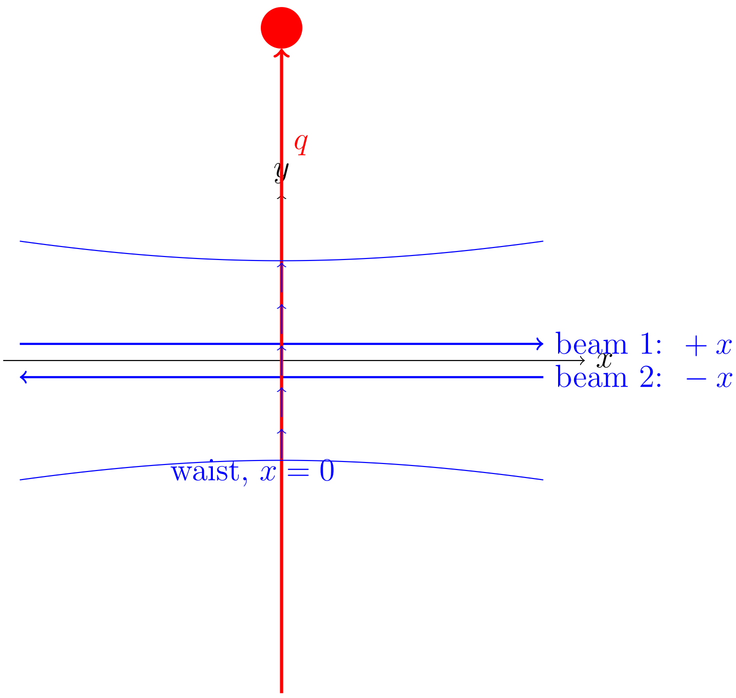
By deriving that energy density with respect to time, and injecting the maxwell equations we get

$$\frac{\partial u_{\text{field}}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E}_{\text{beam}}.\mathbf{E}_{\text{charge}}) + \frac{1}{\mu_0}\frac{\partial}{\partial t}(\mathbf{B}_{\text{beam}}.\mathbf{B}_{\text{charge}}) = -\frac{\nabla\big(\mathbf{E}_{\text{beam}} \times \mathbf{B}_{\text{charge}} +}{\mu_0}$$

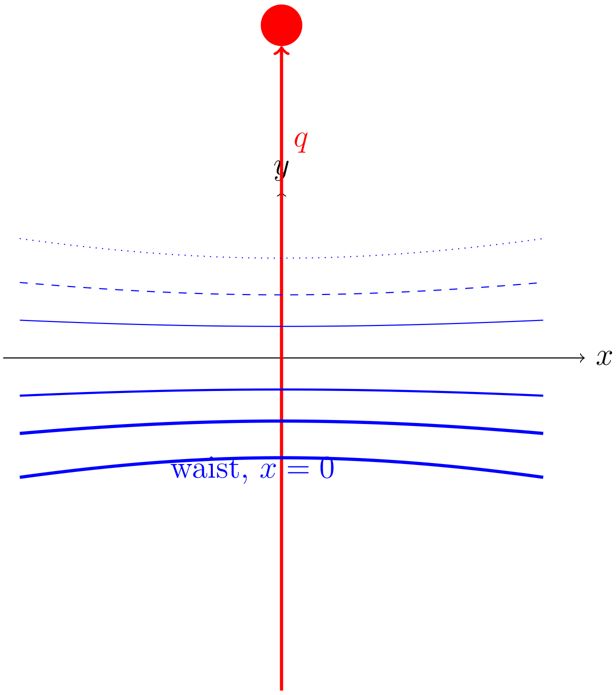
Let's assume that $E_{y\text{beam}} \gg E_{x\text{beam}}$ and $E_{y\text{beam}} \gg E_{z\text{beam}}$, the main term in the scalar product between the two electric field is thus

$$E_{y\text{beam}} E_{y\text{charge}}$$

So let's imagine that the charge acquired some momentum after interacting with the beam, then when the electric fields interact constructively on one half wavelength, they interact destructively on the next half wavelength.

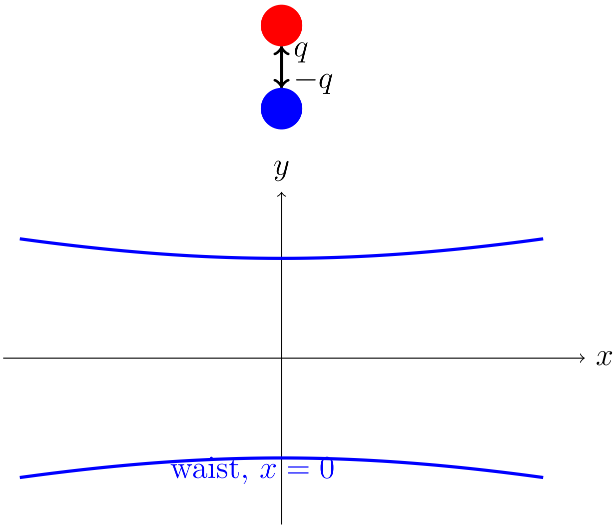


But since some parts of the beam are further from the charges than others, the interferences are stronger for the portions of the field which are closer to the charge than for those distant from it. Overall, if the field energy density was to witness a variation in energy density similar to a "recoil" it would look (on average) like that



Where the field energy density would be smaller (on average) in the positive y half space than it would be on the negative y half space.

But...let's imagine a negative charge following the positive charge at the speed $v + \delta$ where $\delta \ll v$ at a distance $\frac{v\pi}{\omega}$, then the interaction between the negative charge and the beam would provide to the negative charge almost the same impulse, $\Delta p_y = 2qE_0\sqrt{\pi}\,\frac{w_0}{v+\delta}\,e^{-\frac{\omega^2 w_0^2}{4(v+\delta)^2}}$



Since the scalar product is a bilinear form, the scalar product:

$$\mathbf{E}_{\text{beam}}.\left(\mathbf{E}_{\text{q+}}+\mathbf{E}_{\text{q-}}\right)=\mathbf{E}_{\text{beam}}.\left(\mathbf{E}_{\text{q+}}\right)+\mathbf{E}_{\text{beam}}.\left(\mathbf{E}_{\text{q-}}\right)$$

the interferences presented before should follow the same linearity rule with respect to the charges field, and the same is true for the field's momentum, thus, everything should be fine... *but*...Because the negative charge has a speed slightly higher than the positive charge there is a time where the two charges are exactly at the same position and their electric fields completely cancel so there is no longer any interference with the beam field... The electrostatic field completely disappeared, and **so does the field momentum**.

Furthermore once at the same position, if both charges accelerate in the direction opposite to their speed during the same duration they can reverse their speed, while swapping their role between the fastest and the slowest charge. They will then interact again with the beam and a nice tuning of the speed would provide again the same interaction. Opposite charges accelerating at the same position in the same direction with the same magnitude do not radiate, thus no radiation could carry away the momentum nor any imaginable interferences between an hypothetical radiation and the beam.

Thus... That interaction is really a special force, I have no idea where the momentum goes. Either the momentum goes in the hidden momentum of the sources instantaneously, and I would be perfectly happy with that, but it would provide an interaction at arbitrary distance, or that type of beam is not possible and a realistic field forbids such interactions. There could be arguments for that. The electric field is given by the expression:

$$\mathbf{E}_{\text{beam}}=-\boldsymbol{\nabla}\phi-\frac{\partial\mathbf{A}}{\partial t}$$

In the charge reference frame if t_i is an instant before any interaction with the beam far from the sources and t_f an instant after the interaction with the beam, also far from the sources, then:

$$\int_{t_i}^{t_f}\frac{\partial\mathbf{A}}{\partial t}dt=\int_{t_i}^{t_f}\frac{d\mathbf{A}}{dt}dt=\mathbf{A}(t_f)-\mathbf{A}(t_i)=\mathbf{0}$$

so the relevant interaction after integration has to be the result of

$$\mathbf{E}_{\text{beam}}=-\nabla\phi$$

Using the Lienard potentials of the sources, accelerating charges actually create an electric field derived from the scalar potential $\nabla\phi$. Usually that field is completely cancelled by a contribution from the vector potential $\mathbf{A}$ but we decided to not consider that contribution since the integrated contribution of the vector potential is zero. We will compute that contribution later. An interesting aspect is the magnitude of the force. Since it is the result of the product of the charge and the beam field, the force could be arbitrarily large. It has always fascinated me. I often imagined myself lost in space, pushing on the dimmest photons with a very large charge without really interacting with them. Unfortunately, in the real world, that force would be more realistically mega low.

Let's now compute the electric field created by remote moving charges belonging to the beam source. We look at the field created by a single moving charge located at $\mathbf{r'}$ at the retarded time t'. First we need to find the dependance of the retarded time t' variations with respect to local variations.

$$\mathbf{R}'.\mathbf{R}' = \|\mathbf{R}'\|^2 = \|\mathbf{r} - \mathbf{r}'\|^2 = c^2(t - t')^2$$

$$2\mathbf{R}'.(d\mathbf{r} - d\mathbf{r}') = 2c(dt - dt')\|\mathbf{R}'\|$$

$$\mathbf{R}'.d\mathbf{r}' - \mathbf{R}'.d\mathbf{r} = c(dt - dt')\|\mathbf{R}'\|$$

$$dt - \frac{\mathbf{R}'.d\mathbf{r}}{c\|\mathbf{R}'\|} = -\frac{\mathbf{R}'.d\mathbf{r}'}{c\|\mathbf{R}'\|} + dt' = (1 - \mathbf{n}'.\frac{\mathbf{v}'}{c})dt' = (1 - \mathbf{n}'.\boldsymbol{\beta}')dt'$$

$$dt' = \frac{1}{(1 - \mathbf{n}'.\boldsymbol{\beta}')}\left(dt - \frac{\mathbf{R}'.d\mathbf{r}}{c\|\mathbf{R}'\|}\right)$$

$$\frac{\partial t'}{\partial t} = \frac{1}{(1 - \mathbf{n}'.\boldsymbol{\beta}')}$$

$$\frac{\partial t'}{\partial x^i} = -\frac{x_i - x_i'}{c\|\mathbf{R}'\|(1 - \mathbf{n}'.\boldsymbol{\beta}')}$$

The retarded potential of a charge q is

$$\phi = \frac{q}{4\pi\epsilon_0(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}')}$$

So

$$
\begin{aligned}
\frac{\partial \phi}{\partial x^i} &= \frac{\partial}{\partial x^i}\frac{q}{4\pi\epsilon_0 \|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'} \\
&= -\frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\frac{\partial}{\partial x^i}\Big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\Big) \\
&= -\frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\Big(\frac{\partial\|\mathbf{R}'\|}{\partial x^i} - \boldsymbol{\beta}'.\frac{\partial \mathbf{R}'}{\partial x^i} - \mathbf{R}'.\frac{\partial \boldsymbol{\beta}'}{\partial x^i}\Big) \\
&= -\frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\Big(\frac{\partial\|\mathbf{R}'\|}{\partial t'}\frac{\partial t'}{\partial x^i} - \boldsymbol{\beta}'.\frac{\partial \mathbf{R}'}{\partial t'}\frac{\partial t'}{\partial x^i} - \mathbf{R}'.\frac{\partial \boldsymbol{\beta}'}{\partial t'}\frac{\partial t'}{\partial x^i}\Big) \\
&= -\frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\Big(\frac{\partial c(t-t')}{\partial t'}\frac{\partial t'}{\partial x^i} + c\boldsymbol{\beta}'^2\frac{\partial t'}{\partial x^i} - \mathbf{R}'.\dot{\boldsymbol{\beta}}'\frac{\partial t'}{\partial x^i}\Big) \\
&= -\frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\Big(- (c - c\boldsymbol{\beta^2} + \mathbf{R}'.\dot{\boldsymbol{\beta}}')\frac{\partial t'}{\partial x^i}\Big) \\
&= \frac{q}{4\pi\epsilon_0\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)^2}\Big(- (c - c\boldsymbol{\beta^2} + \mathbf{R}'.\dot{\boldsymbol{\beta}}')\frac{x_i - x'_i}{c\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\Big) \\
-\boldsymbol{\nabla}\phi &= \frac{q}{4\pi\epsilon_0(1-\mathbf{n}'.\boldsymbol{\beta}')^3}\Big((1-\beta^2 + \frac{\mathbf{R}'.\dot{\boldsymbol{\beta}}'}{c})\frac{\mathbf{n}'}{\|\mathbf{R}'\|^2}\Big) \\
&= \frac{q}{4\pi\epsilon_0(1-\mathbf{n}'.\boldsymbol{\beta}')^3}\Big(\frac{(\mathbf{n}'.\dot{\boldsymbol{\beta}}')\mathbf{n}'}{c\|\mathbf{R}'\|} + \frac{\mathbf{n}'(1-\beta^2)}{\|\mathbf{R}'\|^2}\Big)
\end{aligned}
$$

The only term decreasing $\approx \frac{1}{R'}$ is the term

$$
\frac{q(\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\mathbf{n}'}{4\pi\epsilon_0 c\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^3}
$$

If the beam is the resulting radiation from many oscillating charges along the y axis in the $x = a$ plane, then that term is usually minimal when the beam field is maximal (leaving aside the aberration phenomenon), and is maximal out of the beam scope, where the electric field is always zero (or almost zero). How is it possible? The reason is that at each instant t, the vector potential creates a term that exactly cancels the electric field created by the scalar potential. Althought the integrated contribution of the vector potential to the momentum transfert is $\mathbf{0}$, it is not the case at each instant. Let's compute the complete contribution of the vector potential to the electric field.

$$
\begin{aligned}
\frac{\partial \boldsymbol{A}}{\partial t} &= \frac{1}{c^2}\frac{\partial}{\partial t}\frac{q\boldsymbol{v}'}{4\pi\epsilon_0(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}')} \\
&= \frac{q}{c^2}\frac{\partial}{\partial t}\Big(\frac{\boldsymbol{v}'}{4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\Big) \\
&= \frac{q}{c^2 4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\Big(\frac{\partial \boldsymbol{v}'}{\partial t} - \frac{\boldsymbol{v}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\frac{\partial\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)}{\partial t}\Big) \\
&= \frac{q}{c^2 4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\Big(\frac{\partial \boldsymbol{v}'}{\partial t'}\frac{\partial t'}{\partial t} - \frac{\boldsymbol{v}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\frac{\partial\big(\|\mathbf{R}'\| - \mathbf{R}'.\boldsymbol{\beta}'\big)}{\partial t'}\frac{\partial t'}{\partial t}\Big) \\
&= \frac{q}{c 4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^2}\Big(\dot{\boldsymbol{\beta}}' - \frac{\boldsymbol{\beta}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\big(-c + c\boldsymbol{\beta}'^2 - \boldsymbol{R'}.\dot{\boldsymbol{\beta}}'\big)\Big) \\
&= \frac{q}{4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^2}\Big(\frac{\dot{\boldsymbol{\beta}}'}{c} + \frac{\boldsymbol{\beta}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\big(1 - \boldsymbol{\beta}'^2 + \frac{\boldsymbol{R'}.\dot{\boldsymbol{\beta}}'}{c}\big)\Big) \\
&= \frac{q}{4\pi\epsilon_0\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^2}\Big(\frac{\dot{\boldsymbol{\beta}}'}{c} + \frac{\boldsymbol{\beta}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')}\big(1 - \boldsymbol{\beta}'^2 + \frac{\boldsymbol{R'}.\dot{\boldsymbol{\beta}}'}{c}\big)\Big) \\
&= \frac{q}{4\pi\epsilon_0(1-\mathbf{n}'.\boldsymbol{\beta}')^3}\Big(\frac{(1-\mathbf{n}'.\boldsymbol{\beta}')\dot{\boldsymbol{\beta}}' + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}}{c\|\mathbf{R}'\|} + \frac{\boldsymbol{\beta}'}{\|\mathbf{R}'\|^2}\big(1-\boldsymbol{\beta}'^2\big)\Big) \\
-\frac{\partial \boldsymbol{A}}{\partial t} &= \frac{q}{4\pi\epsilon_0(1-\mathbf{n}'.\boldsymbol{\beta}')^3}\Big(-\frac{(1-\mathbf{n}'.\boldsymbol{\beta}')\dot{\boldsymbol{\beta}}' + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}}{c\|\mathbf{R}'\|} - \frac{\boldsymbol{\beta}'}{\|\mathbf{R}'\|^2}\big(1-\boldsymbol{\beta}'^2\big)\Big)
\end{aligned}
$$

The vector $\dot{\boldsymbol{\beta}}'$ can be expressed by decomposing it along $\boldsymbol{n'}$ and a vector normal to **n’** in the plane $(\dot{\boldsymbol{\beta}}', \boldsymbol{n'})$:

$$\dot{\boldsymbol{\beta}}' = \big(\boldsymbol{n'} \times \dot{\boldsymbol{\beta}}'\big) \times \boldsymbol{n'} + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{n}'$$

$$-\frac{\partial \boldsymbol{A}}{\partial t} = \frac{q}{4\pi\epsilon_0(1-\mathbf{n}'.\boldsymbol{\beta}')^3}\Big(-\frac{\big(\boldsymbol{n'} \times \dot{\boldsymbol{\beta}}'\big) \times \boldsymbol{n'} + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{n}' - (\mathbf{n}'.\boldsymbol{\beta}')\dot{\boldsymbol{\beta}}' + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}}{c\|\mathbf{R}'\|} -$$

And we can see that the "radiation" term of the electric field created by the scalar potential is cancelled at each time by a term coming from the vector potential. Slightly repeating myself I need to remind the reader that:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

The whole expression is the complete expression at each instant. The term $-\frac{\partial \boldsymbol{A}}{\partial t}$ is irrelevant when its contribution is *integrated* over a long duration, but its *instantaneous* contribution is relevant, while at the same time the term $-\boldsymbol{\nabla}\phi$ is irrelevant at each *instant* t while it is the only relevant contribution when it is *integrated* over a long duration... super funny. Is it clear? Let me rephrase it. The radiating electric field from the scalar potential is always compensated by an electric field created by the vector potential, but the whole interaction a charge would experience by interacting solely with the field created by the vector potential would be zero. Which means that the scalar potential contribution that would happen "outside" the beam actually finds a counterpart in the field created by the vector potential.

$$\begin{aligned}\int_{t_i}^{t_f} -\boldsymbol{\nabla}\phi_{\text{radiated}} dt &= \frac{q}{4\pi\epsilon_0 c}\int_{t_i}^{t_f} \frac{(\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\mathbf{n}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^3} dt \\ &= -\frac{q}{4\pi\epsilon_0 c}\int_{t_i}^{t_f} \frac{\left(\boldsymbol{n'}\times\dot{\boldsymbol{\beta}}'\right)\times\boldsymbol{n'} - (\mathbf{n}'.\boldsymbol{\beta}')\dot{\boldsymbol{\beta}}' + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^3} dt\end{aligned}$$

It looks like a much more complicated expression but the integrand is now actually at its maximum when and where the beam’s intensity is at its maximum. When $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ have the same orientation, (which is the case here)

$$-(\mathbf{n}'.\boldsymbol{\beta}')\dot{\boldsymbol{\beta}}' + (\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\boldsymbol{\beta} = \mathbf{0}$$

and

$$\frac{q}{4\pi\epsilon_0 c}\int_{t_i}^{t_f} \frac{(\boldsymbol{n'}.\dot{\boldsymbol{\beta}}')\mathbf{n}'}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^3} dt = -\frac{q}{4\pi\epsilon_0 c}\int_{t_i}^{t_f} \frac{\left(\boldsymbol{n'}\times\dot{\boldsymbol{\beta}}'\right)\times\boldsymbol{n'}}{\|\mathbf{R}'\|(1-\mathbf{n}'.\boldsymbol{\beta}')^3} dt$$

Which is not really surprising because the numerator of the integrand is the projection of $\dot{\boldsymbol{\beta}}$ along two axis, orthogonal to each other.