Bounded distances of tile vertices from a line: parallels between the Fibonacci and Hat tilings

Pieter Mostert • 7 Dec 2024

These notes are intended to form part of a series on things I've learned or discovered about the Smith aperiodic monotile, and more generally about edge substitutions of polygonal tilings with edge-matching conditions. Up till now I've been posting on Mastodon and The Tiling List (a mailing list for tiling enthusiasts), and I have a few documents on Google Drive, but I'd like to have a coherent exposition in one place that's publicly accessible and supports longer expositions.

At some stage I may expand this post to cover other parallels between the Fibonacci tiling and the Hat aperiodic monotile, but at the moment I just consider the question whether the 'lifted' tiles can be approximated by a line in $\mathbb{R}^2$ or $\mathbb{C}^2$.

The Fibonacci tiling

Define the substitution σ to be the group endomorphism[1] on the free group $\mathcal{F}$ on two elements, a and b, determined by

$$\sigma(a) = ab, \quad \sigma(b) = a.$$

We can define related endomorphisms on the set of right-infinite, left-infinite, and bi-infinite words on the alphabet $\{a, b\}$, using the limit of σ acting on finite approximations. We'll continue to use σ for these endomorphisms.

Here we will only consider words with letters that have positive exponents, but for the Hat tiling, we will need to consider inverses, so we set things up in this context, even though it isn't necessary.

For a bi-infinite sequence of a's and b's, we use a dot to indicate the point in the sequence between the numbers at positions -1 and 0, for example

$$s_0 = \ldots baa.bab \ldots.$$

The **deflation** of such a sequence s is $\sigma(s)$. For example, the deflation of the sequence s_0 is

$$\sigma(s_0) = \ldots aabab.aaba \ldots$$

We then say that s is the *inflation* of $\sigma(s)$.

The n-shift operator δ_n moves a sequence n steps to the right, so for example

$$\delta_1(s_0) = \ldots ba.abab \ldots.$$

A sequence s is a Fibonacci sequence if it is some shift of the deflation of another Fibonacci sequence, in other words, if there exist bi-infinite sequences s_k and shifts $n_k \in \mathbb{Z}$ such that $s_0 = s$ and

$$s_k = \delta_{n_k}(\sigma(s_{k+1}))$$

for all $k \in \mathbb{N}$. Note that we can restrict n_k to lie in the set $\{0, -1\}$.

Each bi-infinite sequence of a's and b's has a geometric representation as a tiling of the real line, using a prototile set with two tiles which are both unit intervals, and where the dot corresponds to 0. An alternative geometric representation can be given by 'lifting' it to a staircase in $\mathbb{R}^2$, where the a-tiles correspond to horizontal segments of unit length, the b-tiles correspond to vertical segments of unit length, and the dot corresponds to $(0, 0)$.

Note that the end-point of the lift of a finite word w is the image of w under the abelianisation map $\mathcal{F} \to \mathbb{Z}^2$ that sends a to $(1, 0)$ and b to $(0, 1)$. If $M : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the abelianisation of σ, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\sigma"] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathbb{Z}^2 \arrow[r, "M"] & \mathbb{Z}^2
\end{tikzcd}$$

so if we draw the image of the lift of a word w under multiplication by M, it will coincide with the lift of $\sigma(p)$ for all prefixes p of w. We can see this in figure 1 , where the grey paths on the right are the images under multiplication by

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

of the lifted words to the left. Here Γ denotes the lift of a word, and the grey dot denotes the origin. Segments have been decorated with arrows, which aren't necessary here, but are included for consistency with figure 3.

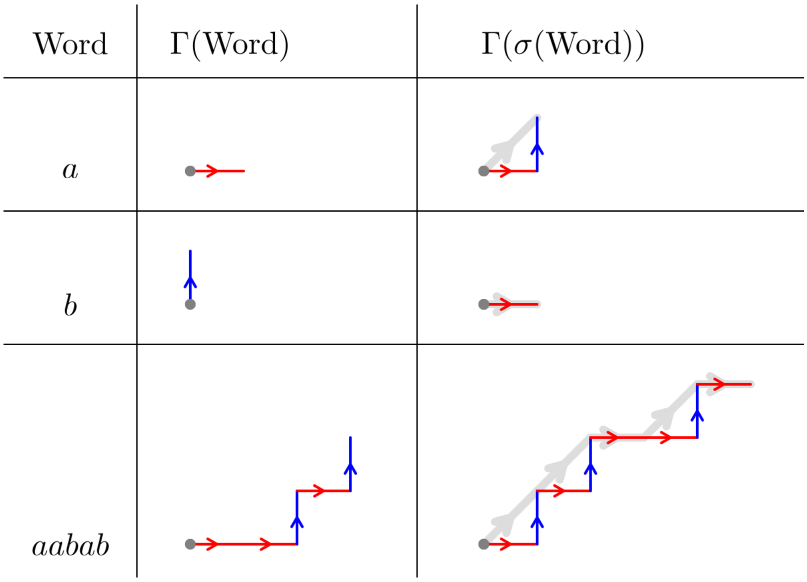


Figure 1: The edge substitution for the Fibonacci deflation on lifted tiles

The tiling of the line associated to a Fibonacci sequence is then equivalent to the orthogonal projection of the associated staircase onto the line $y = x$, up to rescaling. More generally, for $\alpha > 0$, the orthogonal projection of the staircase onto the line $y = \alpha x$ is equivalent to a tiling of the line by two tiles which are intervals whose lengths are in proportion $1 : \alpha$.

Let $\mathcal{V}_0 \subset \mathbb{Z}^2$ be the set of vertices of a lifted Fibonacci tiling. By definition, this is the shift of the deflation of another Fibonacci tiling, so the set of vertices $\mathcal{V}_1 \subset \mathbb{Z}^2$ of the lift of this tiling satisfies

$$M\mathcal{V}_1 + a_0 \subset \mathcal{V}_0$$

for some offset $a_0 \in \mathcal{A} := \{(0,0), (-1,0)\}$. For example:

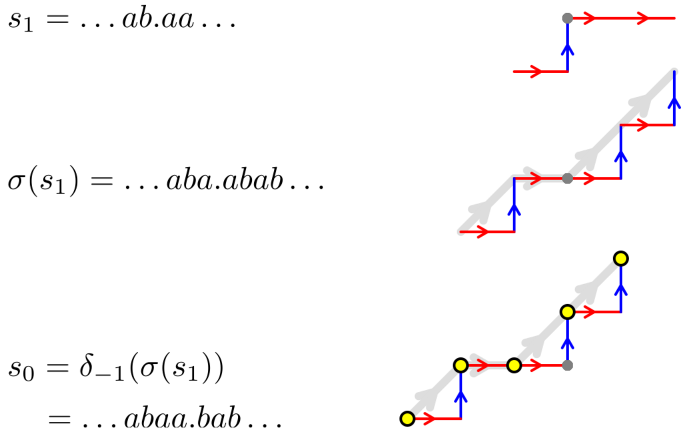


Figure 2: The Fibonacci substitution and shift on lifts of tilings

In the bottom row of figure 2 , the vertices $M\mathcal{V}_1 + a_0$ are shown as yellow dots. A vertex $v_0 \in \mathcal{V}_0$ will either be contained in $M\mathcal{V}_1 + a_0$ or differ from it by $(-1, 0)$. Either way,

$$v_0 + t_0 \in M\mathcal{V}_1 + a_0$$

for some $t_0 \in \mathcal{A}$, so

$$v_0 = Mv_1 + d_0$$

for some $v_1 \in \mathcal{V}_1$ and $d_0 \in \mathcal{A} - \mathcal{A}$.

By iterating this process, for all $k \in \mathbb{N}$ we can find $d_k \in \mathcal{A} - \mathcal{A}$ and v_k in the set of vertices $\mathcal{V}_k$ of lifts of Fibonacci tilings with

$$v_k = Mv_{k+1} + d_k.$$

By induction, for any $n \in \mathbb{N}$,

$$v_0 = M^n v_n + \sum_{k=0}^{n-1} M^k d_k. \tag{1}$$

Let $\lambda_E = \frac{1+\sqrt{5}}{2}$ and $\lambda_C = \frac{1-\sqrt{5}}{2}$ be the eigenvalues of M that have moduli respectively greater and less than 1, and let π_E and π_C be the projections from $\mathbb{R}^2$ onto the respective eigenspaces, which we denote $\mathbb{H}_E$ and $\mathbb{H}_C$ - the expanding and contracting spaces. Thus for all $v \in \mathbb{R}^2$,

$$\pi_E(Mv) = \lambda_E \pi_E(v), \quad \pi_C(Mv) = \lambda_C \pi_C(v).$$

If we apply π_E to (1), we get

$$\pi_E(v_0) = \lambda_E^n \pi_E(v_n) + \sum_{k=0}^{n-1} \lambda_E^k \pi_E(d_k),$$

so

$$\pi_E(v_n) = \lambda_E^{-n} \pi_E(v_0) - \sum_{k=0}^{n-1} \lambda_E^{k-n} \pi_E(d_k).$$

Since $|\lambda_E| > 1$, as $n \to \infty$ the term $\lambda_E^{-n}\pi_E(v_0)$ approaches 0, while the sum on the right remains bounded by a convergent geometric series. Thus there is $R > 0$ (independent of v_0) such that $\pi_E(v_n)$ eventually remains in the ball of radius R about the origin.

Claim 1. For any $R > 0$, there exists $K > 0$ such that $|\pi_E(v_n)| \leq R$ implies $\|v_n\| \leq K$.

This will be proved below, so for now we assume it is true and proceed with the rest of the proof.

Suppose N_0 has been chosen large enough so that for all $n \geq N_0$, $\pi_E(v_n)$ lies in a ball of radius R. By the claim, $\pi_C(v_n)$ lies in a ball of radius K, for some K depending only on R.

Now apply π_C to (1) to get

$$\pi_C(v_0) = \lambda_C^n \pi_C(v_n) + \sum_{k=0}^{n-1} \lambda_C^k \pi_C(d_k).$$

Since $|\pi_C(v_n)|$ and $|\pi_C(d_k)|$ are bounded and $|\lambda_C| < 1$, we can take the limit $n \to \infty$ to deduce that

$$\pi_C(v_0) = \sum_{k=0}^{\infty} \lambda_C^k \pi_C(d_k),$$

which is bounded by

$$\sum_{k=0}^{\infty} |\lambda_C|^k m_0 = \frac{m_0}{1 - |\lambda_C|},$$

where $m_0 = \max\{|\pi_C(d)| \,:\, d \in \mathcal{A} - \mathcal{A}\}$.

Remark 1. This projection depends on the base point of the tiling; if we shifted the sequence, we would get a projection that differs from this one by an additive constant. To define a shift-invariant projection, observe that we can associate to any vertex v_0 a sequence $(a_k, t_k) \in \mathcal{A} \times \mathcal{A}$, with $d_k = a_k - t_k$, where a_k is independent of the shift (since tilings have *unique* inflations). The affine map that sends a vertex v_0 to

$$\sum_{k=0}^{\infty} \lambda_C^k \pi_C(a_k) = \pi_C(v_0) + \sum_{k=0}^{\infty} \lambda_C^k \pi_C(t_k)$$

is then shift-invariant.

Returning to claim 1, note that since the tiles have length at least $m = 1/(\phi^2 + 1)$ and at most $d = \phi/(\phi^2 + 1)$, the ball of radius R around the origin can intersect at most $N = \lfloor 2(R + d)/m \rfloor + 2$ tiles in any tiling, so any vertex v_n can be joined to the origin in $\mathbb{C}^2$ by a path of length at most N, and thus $\|v_n\|$ is bounded by N.

This may seem like a strange way to prove the claim, which follows more directly by noting that the projection of the staircase onto $\mathbb{H}_E$ is a tiling with finite local complexity, but we make the argument above since a modification of it will be needed in the case of the Hat tiling.

In fact, we can construct a situation involving the Fibonacci tiling that requires a similar modification. Fix $\alpha > -1$, and define

$$\pi_* : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_2) \mapsto x_1 + \alpha x_2,$$

which up to rescaling is the orthogonal projection onto the line $y = \alpha x$. Suppose we wanted to prove claim 1 in this context.

Let $\mathscr{A}$ be the set of left endpoints of lifted a tiles, ordered by the order of the tiles in the sequence, and for $R > 0$, let $\mathscr{A}_R$ be those endpoints v with $|\pi_*(v)| \leq R$. Since Fibonacci sequences don't contain adjacent b's, consecutive elements in $\mathscr{A}$ differ by the vector $(1, 0)$ or $(1, 1)$. Therefore the displacement between the π_*-projections of such points is either 1 or $1 + \alpha > 0$. In either case the displacement is positive, so the size of $\mathscr{A}_R$ is bounded by $N_R = \lfloor 2R/\min\{1, 1 - \alpha\} \rfloor + 1$, and each vertex in $\mathscr{A}_R$ is not more than $2N_R + 2$ steps away from the origin.

Each vertex v in the staircase is at most one step from a vertex in $\mathscr{A}$, so if $|\pi_*(v)| \leq R$, then it is not more than $K = 1 + 2N_{R+d_*} + 2$ steps away from the origin, where $d_* = |\alpha|$ is the length of the π_* projection of the lift of b. These bounds can certainly be improved, but we just care about their existence. This completes the proof of the claim in this case.

The Hat tiling

By a ‘Hat’ tiling, we refer to a *family* of tilings described by Smith et al., which contain tilings by ‘chevron’, ‘turtle’, ‘hat’ and ‘comet’ tiles, among infinitely many others. Our point of view is that these are just different projections of a single object - the lift of the tilings.

We will work with the comet-rhomb version of the Hat tiling, a tiling by three polygonal prototiles with coloured edges that must match. This is described in work by Arnaud Chéritat (implicitly), mathBlock and James Smith.

Instead of working with the free group on two letters, we work with the free group $\mathcal{F}$ on twelve letters

$$\mathcal{L} = \{c_0, c_1, c_2, c_3, c_4, c_5, d_0, d_1, d_2, d_3, d_4, d_5\},$$

which correspond to directed edges of the tiles, or equivalently, to directions in the lifting space.

Let C_6 be the cyclic group of order 6; in this context the most natural way to represent it is as the 6th roots of unity:

$$C_6 = \langle \omega \rangle \subset \mathbb{C}^{\times}, \quad \omega = e^{\pi i/3}.$$

Since C_6 acts on $\mathcal{L}$ by

$$\omega \cdot c_k = c_{k+1}, \quad \omega \cdot d_k = d_{k+1}$$

(subscripts taken modulo 6), it acts on $\mathcal{F}$ and its abelianisation, which we can identify with the additive group of $\mathbb{Z}[C_6] \times Z[C_6]$.

The analogue of the Fibonacci substitution is the C_6-equivariant homomorphism defined by

$$\begin{aligned}\sigma(c_0) &= c_3^{-1} d_5 c_2^{-1} c_0 c_4^{-1} \\ \sigma(d_0) &= c_1^{-1}\end{aligned}$$

Figure 3 shows the result of applying σ to the rhomb and even and odd comets. Note that this is intended to represent the $\mathbb{C}^2$-lifts of the tiles, but to draw them we need to chose some (arbitrary) projection to $\mathbb{C}$. The one chosen here corresponds to a tile halfway between the hat and spectre versions.

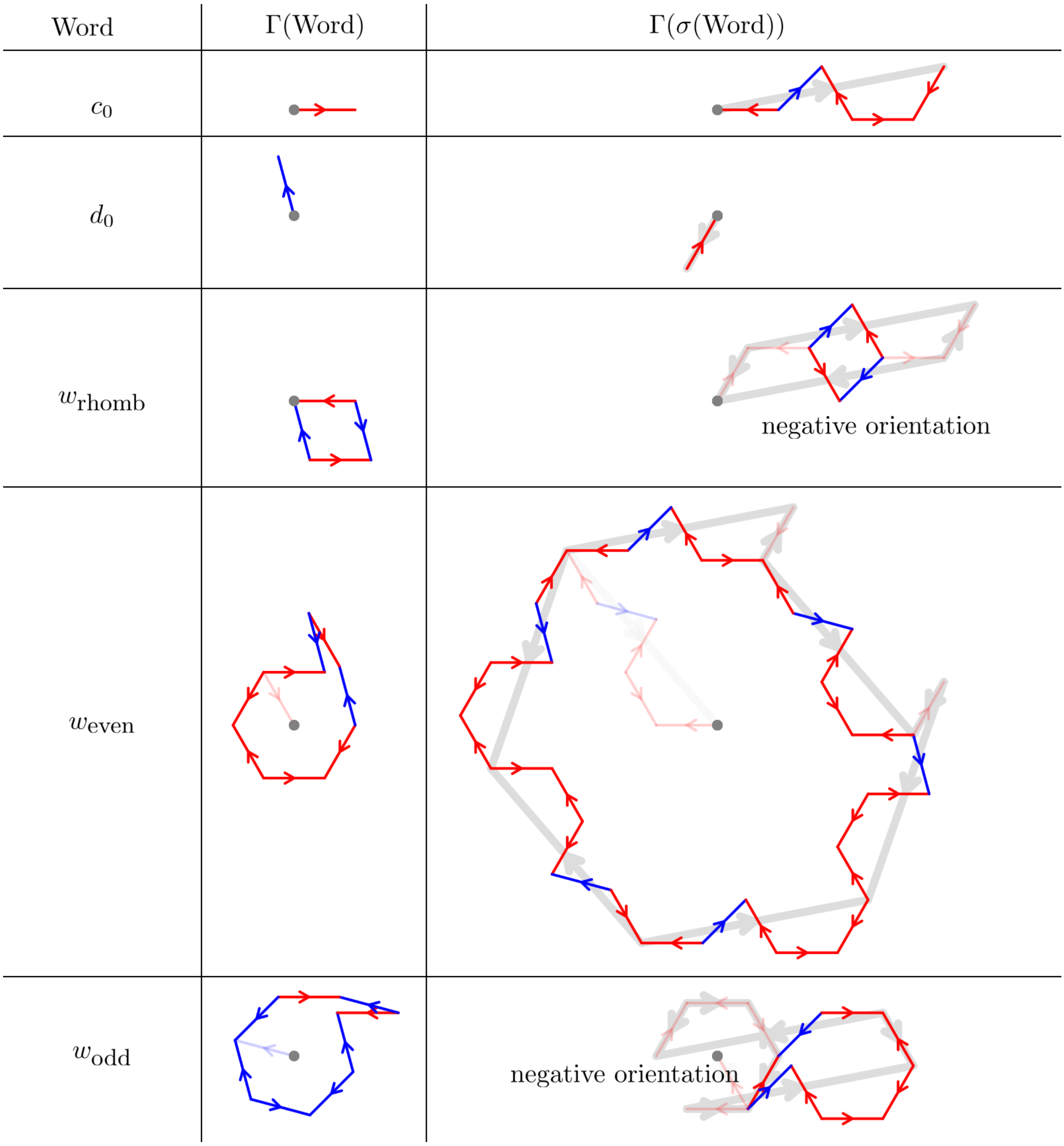


Figure 3: The edge substitution for the Hat deflation on lifted tiles

If c_0 and d_0 correspond to the lifted points $(1, 0)$ and $(0, 1)$ in $\mathbb{C}^2$ respectively, then the analogue of the 2×2 transformation matrix used for the Fibonacci tiling is the matrix

$$M = \begin{pmatrix} 3 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}.$$

Figure 4 shows the vertices $M\mathcal{V}$ (shown in yellow), for $\mathcal{V}$ the vertices of an even tile, and the decomposition of the transformed shape into an H_8 patch.

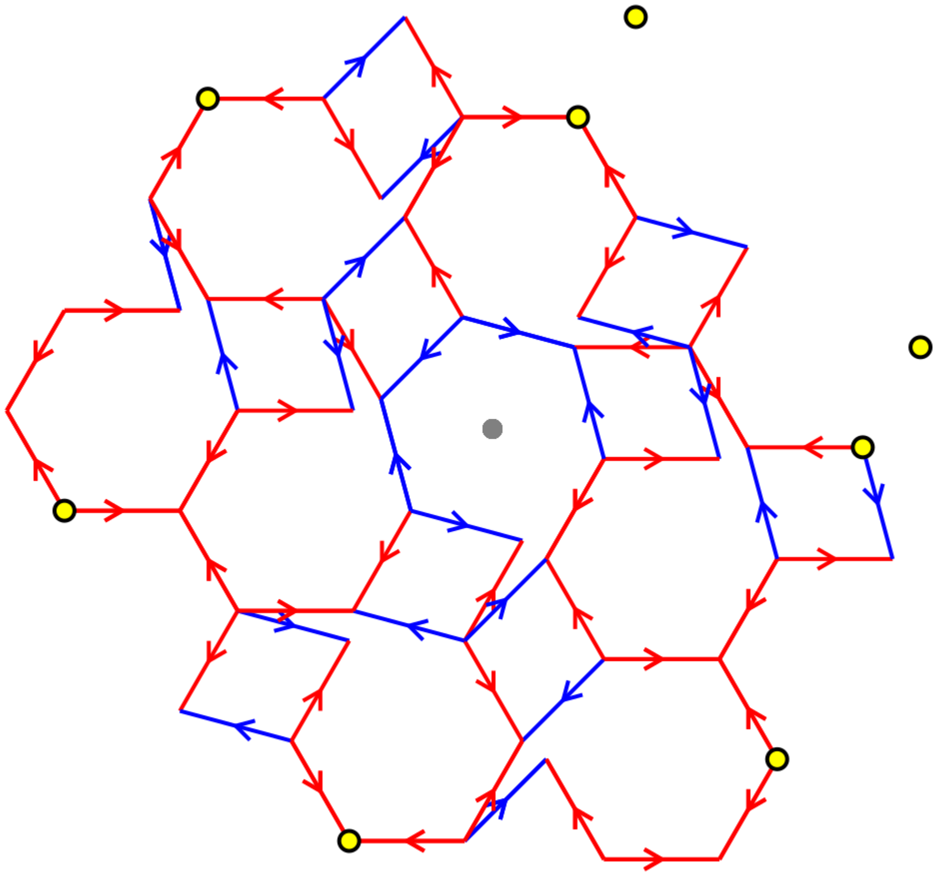


Figure 4: The decomposition of a transformed even tile

Each vertex in the deflated patch differs from some vertex in $M\mathcal{V}$ by an offset coming from a finite set $\mathcal{A}$, so we are in the same situation as the Fibonacci tiling. The proof proceeds as before, with the following differences:

1. $\lambda_E = \frac{3+\sqrt{5}}{2}$ and $\lambda_C = \frac{3-\sqrt{5}}{2}$.
2. $\mathbb{R}$ is replaced by $\mathbb{C}$, so the eigenspaces are *complex* lines.
3. The vertices lie in the product of two copies of the Eisenstein integers $\mathbb{Z}[\omega]$ instead of $\mathbb{Z}^2$.
4. The proof of claim 1 in this context needs to be modified.

Regarding the last point, we will consider the π_E-projections of the centres of lifted even[2] comet tiles, as an analogue of the left endpoints of lifted a-tiles in the Fibonacci tiling.

Consider paths between centres of even tiles in the $\mathbb{C}^2$ lift that consist of moving along straight lines between adjacent points in $\mathbb{Z}[\omega]^2$. By considering cases, each pair of adjacent even tile centres has a connecting path that is some rotation by a power of ω of the ones shown below[3]. Here we show the word associated to each path, as well as its image under the abelianisation map (which gives the coordinate of the end of the path in $\mathbb{C}^2$).

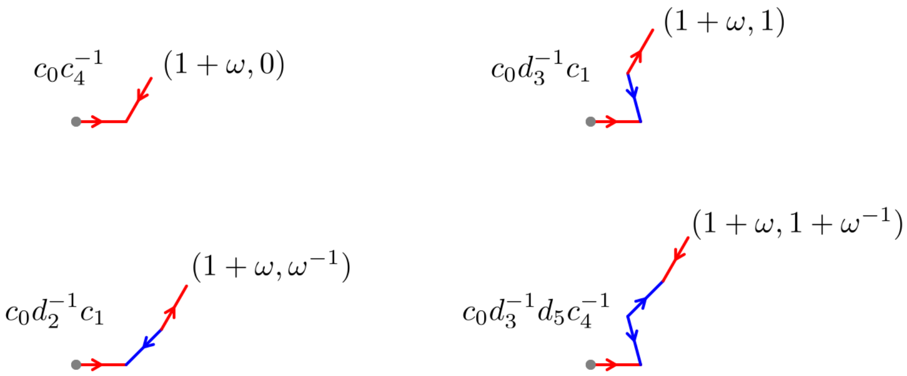


Figure 5: Different possibilities for lifted paths that connect adjacent odd tile centres

Therefore the displacements between adjacent even centres in $\mathbb{C}^2$ are all of the form

$$(1+\omega, n+m\omega^{-1}), \quad n,m \in \{0,1\}, \tag{2}$$

up to multiplication by powers of ω. Thus the modulus of the second coordinate is at most $|1+\omega^{-1}| = \sqrt{3}$.

If we project from $\mathbb{C}^2$ onto $\mathbb{C}$ using the map

$$\pi_1 : (z_1, z_2) \mapsto z_1,$$

the even tiles in the lifted tiling are mapped to tiles of a hexagonal tiling, so their centres form a triangular grid. The projection π_E, up to rescaling, is

$$(z_1, z_2) \mapsto z_1 - \omega\lambda_C z_2,$$

which we can think of as a perturbation of the projection onto the first coordinate, since $|\omega\lambda_C| \approx 0.382$ is smallish. In figure 6 we show various projections[4] of a lifted patch of tiles:

- Left: the projection giving rise to a conventional tiling in the Hat family (the same as in figure 3),
- Centre: the projection under π_1,
- Right: the projection under π_E.

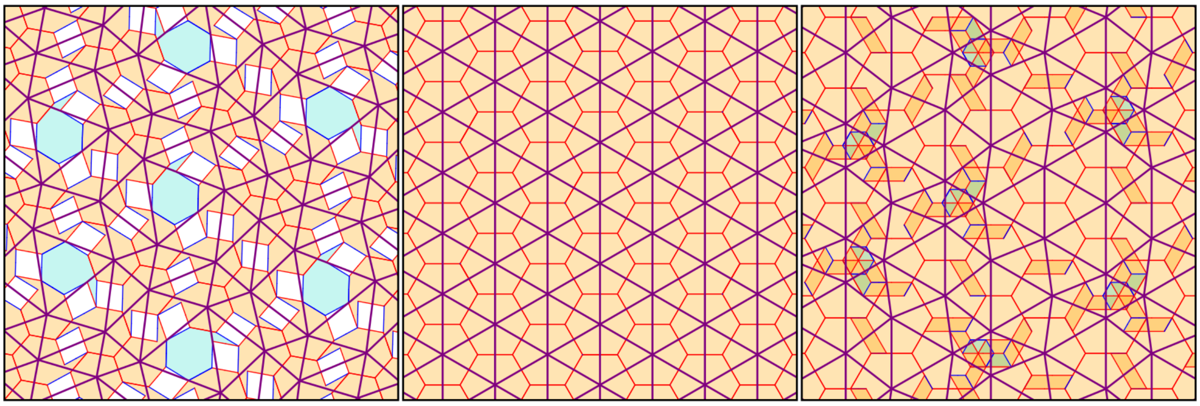


Figure 6: Various projections of a patch of lifted comet-rhomb tiles

The purple segments join centres of adjacent even tiles, and as observed above, form a regular triangular grid in the case of π_1. To show that they still lie on a (deformed) triangular grid for the perturbed projection π_E, consider the triangle in $\mathbb{C}^2$ formed by three mutually adjacent vertices, where the tiling has been translated so that one of them is at the origin.

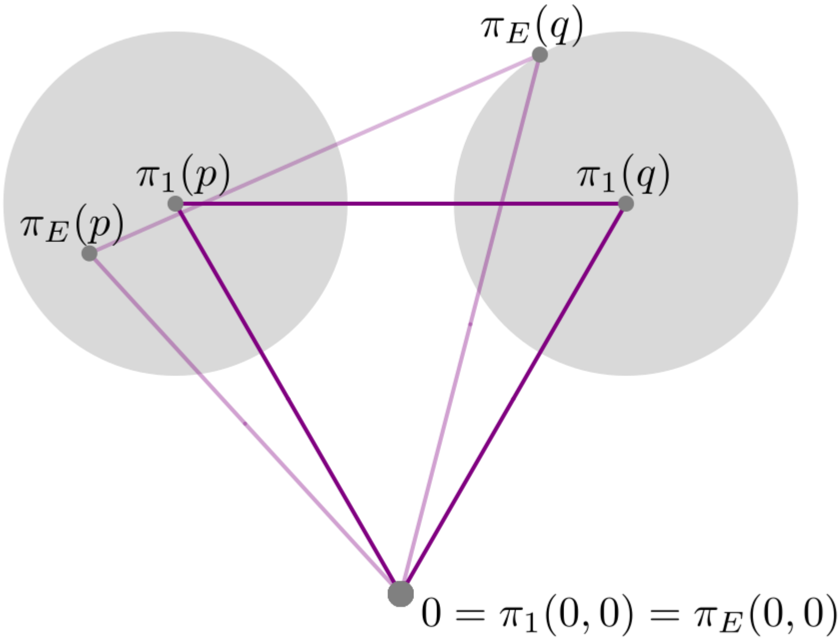


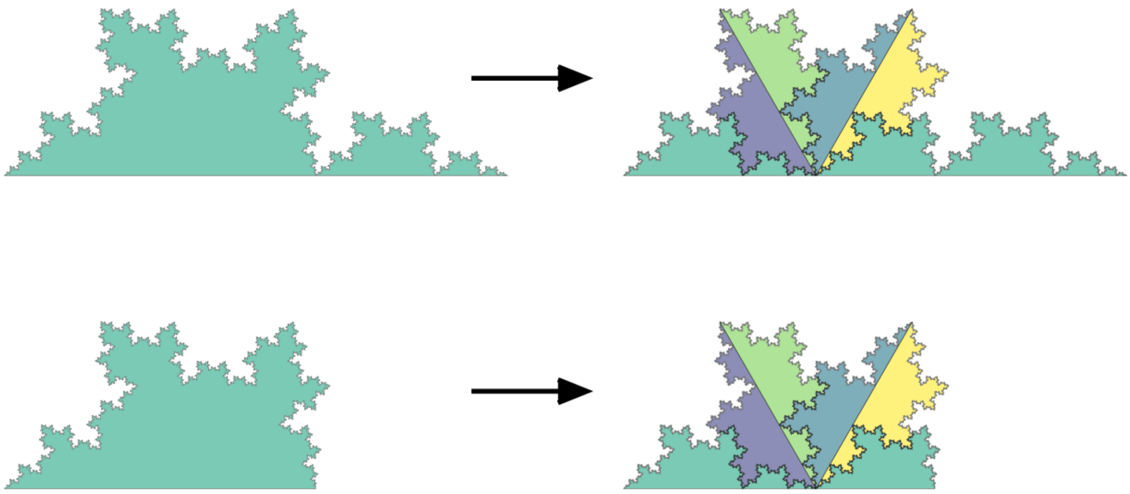
Figure 7: Projections of three mutually adjacent odd tile centres under π_1 and π_E.

Figure 7 shows the projection of this triangle onto $\mathbb{C}$ under π_1 and π_E. From 2, the equilateral triangle has edges of length $|1+\omega| = \sqrt{3}$, while the perturbed projections of vertices ($\pi_E(p)$ and $\pi_E(q)$) have moved a distance of at most $|(1+\omega^{-1})\omega\lambda_C| = \sqrt{3}\lambda_C$ (the radius of the grey circles) from $\pi_1(p)$ and $\pi_1(q)$ respectively. Since $\lambda_C < 1/2$, these circles are disjoint, so none of the triangles can flip orientation, and thus the tiling of the plane by triangles whose vertices lie on even tile centres remains a conventional tiling.

Let $A > 0$ be the minimum area of these triangles. If we subdivide each triangle into three polygons of equal area, we can associate to each even tile centre a neighbourhood of area at least $6(A/3) = 2A$, where neighbourhoods associated to different odd tile centres only overlap on their boundaries.

Consider all the even tile centres in a ball of radius R. If d is the maximum diameter of a triangle, the ball can contain at most $\pi(R+d)^2/(2A)$ even tile centres, since the ball of radius $R+d$ contains all the neighbourhoods of the even tile centres. This completes the central part of the claim, and the rest proceeds as in the proof of the Fibonacci tiling projected onto $y = \alpha x$, $\alpha > -1$.

Remark 2. We can construct a similar representation of the projection of the centre of an even tile (the centre of the red hexagon) onto the contracting subspace as an infinite series with coefficients coming from its position relative to the centre of the H_7 or H_8 patch in which it lies. The shift-invariant version of this projection, together with the inflation decomposition, can be used to construct a graph-directed iterated function system whose solution involves the fractal windows shown below.



Decomposition of fractal windows for the Hat tiling

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1. Easily seen to be an automorphism.↩

2. We call Hat tiles even or odd depending on whether or not they fall in the dominant handedness class.↩

3. Compare this with figure 85 of the preprint by Arnaud Chéritat, which considers the Spectre tiling.↩

4. These have all been rescaled so that tiles are in approximately the same place in each projection.↩