The triply punctured sphere

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I'll develop this post later, but the main purpose for now is to consolidate and share some thoughts/notes of mine that have been floating around for a few semesters. Let $U = \mathbb{C} \setminus \{0,1\}$ denote the doubly punctured plane, or equivalently, the triply punctured Riemann sphere. We will show that the universal cover of U is the upper half-plane.

We have a natural covering map $\lambda : \mathbb{H} \to \mathbb{C}$. To define this, associate to each $\tau \in \mathbb{H}$ a complex torus given by $E = \mathbb{C}/\Lambda_\tau$, where Λ_τ is generated by $1, \tau$. Moreover, fix a basis for the points of order 2 on this quotient; let $e_0 = 0, e_1 = 1/2, e_2 = \tau/2, e_3 = e_1 + e_2$.

The Weierstrass $\wp$-function gives a natural degree-two map

$$\wp : E \to \widehat{\mathbb{C}},$$

and note that $\wp$ is uniquely determined by its evenness (up to composition with a M"{o}bius transformation). Define λ by the cross-ratio

$$\lambda(\tau) = [\wp(e_0), \wp(e_1), \wp(e_2), \wp(e_3)]$$

and note that this modular function λ is invariant under the congruence subgroup $E/\Gamma(2)$. Thus, it descends to an isomorphism $\lambda : \mathbb{H}/\Gamma(2) \to \mathbb{C} \setminus \{0,1\}$. I'll edit this post later to justify some of these facts in greater detail, and I'll also prove Picard's Little Theorem with it.