Holomorphic one-forms and the genus

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In this post, I'll use the Riemann-Roch Theorem to prove that the topological genus and the geometric genus agree. I will work with compact Riemann surfaces, but since these are the same as smooth algebraic curves over $\mathbb{C}$, these results are valid in the algebraic setting as well.

Recall the statement of Riemann-Roch: If X is a compact Riemann surface with genus g, and D is a divisor on X, then

$$\dim(L(D)) - \dim(L(K - D)) = \deg(D) - g + 1.$$

Define the *topological genus* to be the quantity g appearing above, which is intuitively the number of holes that X has (by the classification of compact surfaces and the fact that the complex structure induces orientability, it follows that the underlying real 2-manifold of X is the connected sum of g tori). Define the *geometric genus* to be the dimension of the space of holomorphic 1-forms $\Omega^1(X)$.

Theorem. The geometric genus and topological genus agree.

Proof. If D is the empty divisor, then $L(D)$ is just the space of all holomorphic functions. Now, the space of holomorphic functions on any compact Riemann surface is $\mathbb{C}$ by the maximum principle, i.e. one-dimensional. Putting $D = 0$ into the Riemann-Roch formula, alongside the fact that $\dim L(0) = 1$, we have that $\dim L(K) = 1$. Now, suppose that $K = (\omega)$ for some meromorphic 1-form ω. For any $f \in L(K)$, the 1-form $f\omega$ will not have any poles: if it did, $(f) + (\omega)$ would not be everywhere nonnegative. That is, $f\omega$ is a holomorphic 1-form. Conversely, suppose η is a holomorphic 1-form; we can write $\eta = h\omega$, for some meromorphic function h. Since $(h) + (\omega) = (\eta) \geq 0$, it follows that $h \in L(K)$. This gives a linear isomorphism between $\Omega^1(X)$ and $L(K)$, and since $L(K)$ is g-dimensional, so too is $\Omega^1(X)$. $\square$

There is a proof using Hodge theory as well: we can decompose $H^1(X) = H^{1,0}(X) \oplus H^{0,1}(X)$ into holomorphic and antiholomorphic forms respectively. Since $H^1(X)$ is $2g$-dimensional, the space of holomorphic 1-forms $\Omega(X) = H^{1,0}(X)$ is g-dimensional.