# Conformal mappings between annuli

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In this post, I wanted to share a neat proof that conformally isomorphic annuli have common radial ratio. There are a number of ways to go about this, but I've always found this technique particularly striking, as it relates to the broader theory of conformal invariants and extremal length.

Suppose we have a biholomorphic map $f : A(1, R) \to A(1, S)$, where $R, S > 1$. By inverting if necessary, we can assume $f$ maps the inner circle to the inner circle and maps the outer circle to the outer circle. Expand $f$ as a Laurent series in the annulus, say

$$f(z) = \sum_{n \geq -N} a_n z^n$$

The area bounded by a Jordan curve $\gamma$ is

$$\text{Area}(\gamma) = \frac{1}{2i} \int_\gamma \overline{z} dz.$$

Then

$$\text{Area}(f(S^1(t))) = \frac{1}{2i} \int_{f(S^1(t))} \overline{z} dz = \frac{1}{2i} \int_0^{2\pi} \overline{f(te^{i\theta})} f'(te^{i\theta}) tie^{i\theta} d\theta$$

We can plug in Laurent series for $f$ and that of its derivative to get

$$\text{Area}(f(S^1(t))) = \frac{1}{2i} \int_0^{2\pi} \overline{f(te^{i\theta})} f'(te^{i\theta}) tie^{i\theta} d\theta.$$

$$= \pi \sum_{n \geq -N} n|a_n|^2 t^{2n}$$

This holds *a priori* for $t \in (1, R)$, but by continuity, it also holds for $t = 1$ and $t = R$. If we set $t = 1$, we get

$$\pi = \pi \sum_{n \geq -N} n|a_n|^2$$

If we set $t = R$, we get

$$\pi R^2 = \pi \sum_{n \geq -N} n|a_n|^2 R^{2n}$$

We can combine these to find that

$$\pi S^2 - \pi R^2 = \pi R^2 \sum_{n \geq -N} n|a_n|^2(R^{2n-2} - 1)$$

Regardless of whether $n$ is positive, negative, or zero, each of the terms $n|a_n|^2(R^{2n-2} - 1)$ is nonnegative!

This shows that $S \geq R$. Applying the same argument with $f$ replaced by $f^{-1}$ shows that $R \geq S$, so $R = S$. $\square$