# Birational invariance of plurigenera

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In this post, we introduce quantities called *plurigenera* that are associated to smooth, complete varieties, and show that they are birational invariants. We offer two sources of motivation:

1. *Surfaces.* The classification of curves essentially amounts to stating that there exists one invariant (the genus), and for curves of fixed genus $g$, there is a $(3g-3)$-dimensional family. Surfaces, however, are more complicated, and we anticipate their moduli to have more intricate data than in the case of curves; they cannot be classified by a single invariant. Without getting too far afield into a "complete" taxonomy (due to Enriques-Kodaira), we wish to find invariants of surfaces, of which the plurigenera will serve as a family of examples that straightforwardly generalize the genus of a curve. Moreover, they generalize to higher-dimensional $n$-folds.

2. *Rationality.* A question that remained open for some time was whether unirational varieties were rational – that is, if varieties admitting a dominant rational map from projective space were actually birationally equivalent to projective space. For curves, this was answered in the affirmative by a classical theorem of Luroth, and for surfaces, a result of Castelnuovo also indicated its verity. However, numerous counterexamples emerged for threefolds in the 70s. A general program, then, was to find birational invariants that took on different values for a given example when compared to $\mathbb{P}^n$.

**Definition.** Let $X$ be a smooth, complete variety over a field $k$. The *plurigenera* of $X$ are

$$p_m(X) := \dim_k H^0(X, \omega_X^{\otimes m}).$$

Recalling that one definition for the genus of a curve $C$ was the dimension of the space of 1-forms on it, we recover $g_C = p_1(C)$.

**Theorem.** For all $m \geq 1$, the plurigenera $p_m(X)$ are birational invariants for smooth complete varieties.

Hartshorne proves that the genus of a curve is a birational invariant, a proof which we shall adapt below.

*Proof.* Let $f : X \dashrightarrow Y$ be a birational map of smooth complete varieties. Since $X$ is normal and $Y$ is complete, $f$ extends to a morphism $f : U \to Y$, where $U \subseteq X$ is open and such that $\operatorname{codim}_X(X \setminus U) \geq 2$. To compare forms on $X$ to those on $U$, we have a pullback morphism

$$f^*\Omega^1_Y \to \Omega^1_U$$

which is an isomorphism on the open subset of $U$ over which $f$ is an isomorphism. Then, taking top exterior powers and $m$th tensor powers induces a morphism

$$f^*(\Omega^n_Y)^{\otimes m} \to (\Omega^n_U)^{\otimes m},$$

which is still generically an isomorphism. This is an injection of sheaves since both are locally free, hence torsion-free. All of this gives rise to the following commutative diagram:

$$\begin{array}{ccc} H^0(U, f^*(\Omega^n_Y)^{\otimes m}) & \hookrightarrow & H^0(U, (\Omega^n_U)^{\otimes m}) \\ \uparrow & & \uparrow \text{res} \\ H^0(Y, (\Omega^n_Y)^{\otimes m}) & \hookleftarrow & H^0(X, (\Omega^n_X)^{\otimes m}) \end{array}$$

where the left-hand is injective since $f$ is dominant, and the right-hand is an isomorphism since $\operatorname{codim}_X(X \setminus U) \geq 2$, $(\Omega^n_X)^{\otimes m}$ is locally free, and since $X$ is normal (and so $S_2$, i.e. functions extend over codimension two or greater). Taking dimensions, it follows that $p_m(Y) \leq p_m(X)$ for all $m \geq 1$; replacing $f$ with the inverse of the birational map, we get that $p_m(X) \leq p_m(Y)$, which proves that $p_m(X) = p_m(Y)$ for all $m > 0$, as desired. $\square$