The Tate curve and p-adic analytic geometry, part 1

akrishna168 • 2 Jan 2025

In this post, we will discuss (without any proofs) Tate's parameterization of elliptic curves over p-adic fields. Aside from intrinsic importance, his work marked the beginning of $\textit{rigid analytic geometry}$, which I'll likely discuss in future posts.

Recall that an elliptic curve E over a field K is a smooth, irreducible projective curve of genus 1 with a choice of distinguished point O. An elliptic curve has a group law (in fact, $E \cong \mathrm{Pic}^0(E)$), for which the point O is the identity; over $\mathbb{C}$, this is the same as a Riemann surface of topological genus 1 equipped with a choice of point. Weierstrass was able to $\textit{uniformize}$ elliptic curves over $\mathbb{C}$: by starting with their description as a complex torus $\mathbb{C}/\Lambda$, where $\Lambda \subseteq \mathbb{C}$ is a lattice of rank two, he used the $\wp$-function (which expresses a complex torus as a branched, two-sheeted cover of the Riemann sphere $\widehat{\mathbb{C}}$ at the group of four 2-torsion points $E[2]$) to realize $\mathbb{C}/\Lambda$ as a cubic in $\mathbb{P}^2$ via the map $[\wp : \wp' : 1]$. This renders more explicit the shape of the n-torsion subgroups, the group law, and the possible endomorphism rings.

Tate wanted to mimic this construction for elliptic curves over $\mathbb{Q}_p$, but immediately ran into a problem: the p-adic numbers have no nontrivial lattices! To see this, let $\Lambda \subseteq \mathbb{Q}_p$ be a nonzero subgroup. Take $t \in \Lambda$, not zero, and note that $p^n t \in \Lambda$ for all $n \geq 0$. Moreover, $\lim_{n\to\infty} p^n t = 0$, so Λ accumulates at 0, contradicting discreteness.

To mitigate this, we can apply the exponential in the complex setting to obtain an alternative description of elliptic curves as those of the form $\mathbb{C}^*/q^{\mathbb{Z}}$. This topologically checks out, although it is slightly harder to see – the space $\mathbb{C}^*$ is biholomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$, and then modding out by powers of q under the homomorphism $\exp$ effects quotienting by another rank-one lattice. Thus, these successive quotients recover the description of elliptic curves as $\mathbb{C}$ modulo a rank-two lattice. Formally mimicking this construction bodes much better for $\mathbb{Q}_p$, since the group $\mathbb{Q}_p^*$ has plenty of discrete subgroups; for example, any $q \in \mathbb{Q}_p^*$ with $|q| < 1$ defines the discrete subgroup $q^{\mathbb{Z}}$. We will use this description to furnish a p-adic analytic isomorphism between $\mathbb{Q}_p^*$ with a p-adic elliptic curve E_q, and subsequently obtain a similar parameterization to that in the complex case.

All of this is summarized in the following theorem: **Theorem.** (Tate) Let K be a finite extension of $\mathbb{Q}_p$. Let $q \in K^*$ satisfy $|q| < 1$, and define

$$s_k(q) = \sum_{n\geq 1} \frac{n^k q^n}{1-q^n}, a_4(q) = -5s_3(q), a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}.$$

a. The series $a_4(q), a_6(q)$ converge in K, enabling us to define the $Tate\ curve$ by the equation

b. $$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

   The Tate curve is an elliptic curve over K with discriminant and j-invariant given by

c. $$\Delta(E_q) = q\prod_{n\geq 1}(1-q^n)^{24}, j(E_q) = \frac{1}{q} + 744 + 196884q + \cdots$$

   The series

$$X(u,q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q), Y(u,q) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q)$$

   converge for all $u \in \bar{K} \setminus q^{\mathbb{Z}}$. They define a surjective morphism

$$\phi : \bar{K}^* \to E_q(\bar{K})$$

   sending $u \mapsto (X(u,q), Y(u,q))$ for $u \notin q^{\mathbb{Z}}$, and $u \to O$ if $u \in q^{\mathbb{Z}}$.

d. The morphism ϕ is compatible with the action of the absolute Galois group $G_K := \mathrm{Gal}(\bar{K}/K)$, namely,

$$\phi(u^\sigma) = \phi(u)^\sigma$$

   for all $u \in \bar{K}^*, \sigma \in G_K$.

The arithmetic contribution from the last part of the theorem, absent in the theory of elliptic curves over $\mathbb{C}$, tells us that for any algebraic extension L/K, we have an isomorphism

$$\phi : L^*/q^{\mathbb{Z}} \to E_q(L).$$

Our programme is as follows: we will flesh out the story of elliptic curves over $\mathbb{C}$ to indicate where the formulas in Tate's result come from, and then prove each part of the p-adic version. After that, we will indicate some interpretations, as well as an application to the theory of complex multiplication.