The canonical sheaf of projective space

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Let X be a smooth variety of dimension n, with sheaf of differentials Ω_X. Define the *canonical sheaf* to be the line bundle $\omega_X := \bigwedge^n \Omega_X$, whose sections are differential forms. The class of a divisor of a form is the *canonical class* K_X.

In this post, we will calculate the canonical sheaf of projective space $\mathbb{P}^r$. In general, one can define the canonical sheaf for any projective scheme, in which case one usually refers to it as “dualizing”. Such calculations often reduce to the case of projective space, so the following computation is the most central one.

Theorem. $\omega_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r-1)$.

Proof. Let $x_0, \ldots, x_r$ be homogeneous coordinates on $\mathbb{P}^r$. Let H be the hypersurface defined by the vanishing of the coordinate x_0, and let $U = \mathbb{P}^r \setminus H$ be its complement; note that with the coordinates $z_i = x_i/x_0$, we have an identification $U \cong \mathbb{A}^r$.

On U, then, the space of differential r-forms is spanned by $dz_1 \wedge \cdots \wedge dz_r$. By properties of differentials, the “formal quotient rule” holds, and we can express

$$d\left(\frac{x_i}{x_0}\right) = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

which implies that

$$d\left(\frac{x_1}{x_0}\right) \wedge \cdots d\left(\frac{x_r}{x_0}\right) = \frac{dx_1 \wedge \cdots \wedge dx_r}{x_0^r} - \sum_{i=1}^{r} x_i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r}{x_0^{r+1}},$$

which has a pole of order $(r+1)$ on U.

It follows that $\operatorname{div} \omega = -(r+1)H$, which is the canonical class. $\square$