Singular plane curves in pencils, topologically

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Introduction

Suppose we have a pencil $\{C_t = V(t_0F + t_1G) \subseteq \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ of plane curves of degree d – how many are singular? In this note, we discuss a purely topological approach to this enumerative problem, which is juxtaposed against the “usual” technique involving Chern classes. Throughout, for simplicity, we assume our base field is $k = \mathbb{C}$ (by the Lefschetz Principle, our discussion carries over to all algebraically closed fields of characteristic zero.)

The Classical Riemann-Hurwitz Formula

As a warmup, we start with the Riemann-Hurwitz formula. Let X be a smooth curve of degree g, and let $f : X \to C$ be a branched cover of degree d. We want to determine a relationship between the Euler characteristics of the curves; to do this, we start with the simplest case of an unramified cover. Purely topologically, if $X \to C$ is a d-fold covering space, then their Euler characteristics satisfy

$$\chi(X) = d \cdot \chi(C),$$

as the Euler characteristic is additive for disjoint unions. When ramification enters the picture, we modify the above formula with an error term:

$$\chi(X) = d \cdot \chi(C) + \sum_{p \in X} (e_x - 1)$$

where e_p is the ramification index at the point p.

The Generalized Riemann-Hurwitz Formula

To generalize Riemann-Hurwitz, we can think of it as a relationship between the topological Euler characteristics of a covering space and a base, up to some contribution from ramification behavior. More generally, though, for a fiber bundle $E \to X$ with fiber F, we have the following relation:

$$\chi(E) = \chi(X) \cdot \chi(F),$$

and we recall that covering spaces are simply fiber bundles with discrete fibers. Thus, we may approach a form of Riemann-Hurwitz that applies to morphisms that are fiber bundles away from finitely many points.

Let X be a smooth, projective variety, and let $f : X \to C$ be a map to a smooth curve of genus g. Since we are in characteristic 0, there exist finitely many points $p_1, \dots, p_\delta$ such that the fiber X_{p_i} is singular. Let Y be the divisor of X given by the (disjoint) union of these fibers, i.e.

$$Y := \bigcup_{i=1}^{\delta} X_{p_i}.$$

Then, $\chi(Y) = \sum_{i=1}^{\delta} \chi(X_{p_i})$, and since the open set $X \setminus Y$ is a fiber bundle over $C \setminus \{p_1, \dots, p_\delta\}$, we have that

$$\chi(X \setminus Y) = \chi(X_\eta) \cdot \chi(C \setminus \{p_1, \dots, p_\delta\}) = (2 - 2g - \delta) \cdot \chi(X_\eta),$$

where η is a general point of X.

By additivity of the Euler characteristic on disjoint unions, we have that

$$\chi(X) = \chi(X \setminus Y) + \chi(Y)$$

which by the previous calculations, we can rewrite as

$$\chi(X) = (2 - 2g - \delta) \cdot \chi(X_\eta) + \sum_{i=1}^{\delta} \chi(X_{p_i}).$$

Rephrasing the first right-hand summand and extending the latter over all points of C, we get the generalized Riemann-Hurwitz formula:

$$\chi(X) = \chi(X_\eta) \cdot \chi(C) + \sum_{p \in C} (\chi(X_p) - \chi(X_\eta)).$$

That is, we get the expected relation of if X were a true fiber bundle over C, with an error term that counts the deviation of the Euler characteristic of each singular fiber from that of a generic one.

Application to Our Enumerative Problem

Let us apply the generalized Riemann-Hurwitz formula to our enumerative problem from the Introduction. Recall the setup:

$$\{C_t = V(t_0 F + t_1 G) \subset \mathbb{P}^2\}$$

is a general pencil of plane curves of degree d. Because we assumed that the polynomials F and G were general, the base locus

$$\Gamma = V(F, G)$$

of the pencil is d^2 reduced points, and the total space of the pencil, expressed as the graph

$$X = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid p \in C_t\}$$

of the rational map $[F, G] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, is the blowup of $\mathbb{P}^2$ along Γ. Now, X is smooth, so $f : X \to \mathbb{P}^1$ is simply projection onto the first factor.

Since X is the blow-up of $\mathbb{P}^2$ at d^2 points, we have

$$\chi(X) = \chi(\mathbb{P}^2) + d^2 = d^2 + 3.$$

Next, we know that a general fiber C_η of the map f is a smooth plane curve of degree d; by the degree-genus formula, it has genus $g = \frac{(d-1)(d-2)}{2}$, so

$$\chi(C_\eta) = -d^2 + 3d.$$

Now, each singular fiber C has a single node as singularity – no more, no worse. Therefore, the normalization $\widetilde{C}$ of each nodal curve has genus $\frac{(d-1)(d-2)}{2} - 1$, so it has Euler characteristic $\chi(\widetilde{C}) = -d^2 + 3d + 2$. We can view C as coming from $\widetilde{C}$ by identifying two points that form the node, which bumps down the Euler characteristic by one, that is,

$$\chi(C) = -d^2 + 3d + 1,$$

which is 1 greater than the Euler characteristic of a general fiber. Thus, the contribution of each singular fiber to the generalized Riemann-Hurwitz formula is 1, and we can count the number of singular fibers by the following calculation:

$$\delta = \chi(X) - \chi(\mathbb{P}^1) \cdot \chi(C_\eta) = d^2 + 3 - 2(-d^2 + 3d) = 3d^2 - 6d + 3,$$

which is the number we want!

References

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[Ha77] Hartshorne, R. (1977). Algebraic geometry (Vol. 52). Springer-Verlag.