4-Manifolds, Part 1: The intersection form

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This is the first part in a series that I'm doing on 4-manifolds, with an eye toward Donaldson's Diagonalization Theorem. I'll retro-edit this post to contain more introduction and discussion, but for now I'll introduce the intersection form on a 4-manifold, a central invariant. We'll go on to discuss the algebraic topology of smooth 4-manifolds.

Definition. The intersection form of a compact, oriented 4-manifold X is the symmetric bilinear form

$$H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}, \quad (\alpha,\beta) \mapsto (\alpha \cdot \beta) := \langle \alpha \smile \beta, [X] \rangle.$$

By Poincare duality, the induced form on $H^2(X;\mathbb{Z})/\operatorname{tors}$ is nondegenerate, and we sometimes denote it by Q_X.

In general, what can we say about the structure of the homology and cohomology groups of X? Because we are dealing with an oriented manifold, $H_4(X) = H^0(X) \cong \mathbb{Z}$, and $H_0(X) = H_4(X) \cong \mathbb{Z}$. By Hurewicz, we have $H^3(X) = H_1(X) \cong \pi_1^{\mathrm{ab}}$, and we also have $H_3(X) = H^1(X) \cong \operatorname{Hom}(\pi_1(X), \mathbb{Z})$. Moreover, $H_2(X) = H^2(X)$. Finally, by the Universal Coefficient Theorem for cohomology, we have $H^2(X)_{\mathrm{tors}} \cong H_1(X)_{\mathrm{tors}}$.

Imposing the further restriction that X be simply connected, i.e. $\pi_1(X) = 0$, we have that odd-dimension (co)homology vanishes, and $H_2(X) \cong H^2(X)$ is torsion-free.

We will give two further interpretations of the intersection form in subsequent post: (1) via intersections of embedded surfaces, and (2) via differential forms.