Alexander Duality: Part 1

written by akrishna168 on Functor Network
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In this post, I'll talk about Alexander Duality, one of my favorite results in algebraic topology. I'll defer the proof to a subsequent blog post; in this one, I'll state the theorem and discuss some immediate consequences of interest.

The setup is as follows: let $X \subseteq S^n$ be a nonempty, compact subset of the n-sphere such that there exists an open neighborhood $N(X)$ which deformation retracts to X, and such that $\overline{N(X)} \subseteq S^n$ is an n-manifold with boundary.

Theorem (Alexander duality) $\widetilde{H}_*(X) \cong \widetilde{H}^{n-*-1}(S^n \setminus X)$

1. Let's describe a consequence that's of interest in low-dimensional topology. **Example. (Knot complements)** Let X be a *knot*, i.e. the image of an embedding $S^1 \hookrightarrow S^3$. Then, $N(X)$ is a solid torus in S^3 that deformation retracts to S^1, and it is a 2-manifold with boundary $S^1 \times S^1$.

   What are the (reduced) homology groups of X? By Alexander duality, we can write
   $$\begin{aligned}\widetilde{H}_0(X) &= 0 = \widetilde{H}^2(S^3 \setminus X)\\ \widetilde{H}_1(X) &= \mathbb{Z} = \widetilde{H}^1(S^3 \setminus X)\\ \widetilde{H}_2(X) &= 0 = \widetilde{H}^0(S^3 \setminus X)\end{aligned}$$
   in particular, homology cannot tell knot complements apart! More generally, one can consider disks or spheres in S^n; this case has an easier proof than the general procedure that one goes through to prove Alexander duality.

2. Next, we describe a corollary of Alexander duality that places certain restrictions on embeddings into Euclidean space.

   Corollary. If $X \subseteq \mathbb{R}^n$ is compact and admits a neighborhood $N(X)$ as in the statement of Alexander duality, then $H_i(X) = 0$ for $i \geq n$ and torsionfree for $i = n-1$ and $n-2$.

   Proof. We can view $\mathbb{R}^n$ (and so X) as a subspace of the one-point compactification S^n, in which case Alexander duality provides isomorphisms $H_i(X) \cong H^{n-i-1}(S^n \setminus X)$. Now, the right-hand group is zero for $i \geq n$ and it is torsionfree for $i = n-1$. Since X has finitely generated homology groups, we may apply the Universal Coefficient Theorem for Cohomology to deduce the result. □

   Consequently, it is impossible to embed a closed, nonorientable n-manifold M into $\mathbb{R}^{n+1}$ for $n > 1$, because $H_{n-1}(M)$ has a copy of $\mathbb{Z}/2$. This implies, for instance, that one cannot embed the Klein bottle into $\mathbb{R}^3$.

3. Now, we prove the (generalized) Jordan Curve Theorem, whose innocuous statement eluded mathematicians for ages until its eventual proof in 1887.

   Theorem. Let X be an n-dimensional *topological sphere* in $\mathbb{R}^{n+1}$ i.e. the image of an continuous injection of S^n into Euclidean space of dimension $n+1$. Then, $\mathbb{R}^{n+1} \setminus X$ consists of exactly two path-components.

   The classical case of $n = 1$ corresponds to the fact that every closed curve in the plane creates an interior and an exterior, with the common boundary being the curve itself. The generalization asserts precisely the analogous fact in higher dimensions.

   Proof. Identify S^n with its image X in $\mathbb{R}^{n+1}$, which itself sits inside the one-point compactification S^{n+1}. By Alexander duality, we have

$$\mathbb{Z} \cong \widetilde{H}_n(S^n) \cong \widetilde{H}^0(S^{n+1} \setminus X)$$

   so since ordinary homology has an extra $\mathbb{Z}$-summand in degree zero, we have

$$\widetilde{H}^0(S^{n+1} \setminus X) \cong \mathbb{Z}^2,$$

   i.e. $S^{n+1} \setminus X$ has two path components. $\square$.