Cartier divisors and the Picard group

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This is my first post, so I'll just write about something I've been thinking about as a pilot for my blog. Let's talk about the relationship between (Cartier) divisors and the Picard group of line bundles up to isomorphism.

Let X be an integral scheme with structure sheaf $\mathcal{O}_X$; for all intents and purposes, one can imagine X to be a variety in this post. Let $\mathcal{K}_X$ be the sheaf of rational functions on X, and let $\mathcal{K}_X^*$ be the subsheaf of invertible rational functions. By examining some sheaf-theoretic properties of $\mathcal{K}_X$, we will show that the divisor class group $\mathrm{CaCl}(X)$ is isomorphic to $\mathrm{Pic}(X)$.

First, we have an exact sequence of sheaves of abelian groups on X:

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X* \to 0$$

which induces a long exact sequence on cohomology. Looking at a portion of this sequence, we have

$$\Gamma(X, \mathcal{K}_X^*) \xrightarrow{\phi} \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_X^*).$$

Note that the cokernel of the map ϕ on global sections is by definition the group $\mathrm{CaCl}(X)$ of Cartier divisors on X, while $H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$, as is well-known.

What about the last term? It turns out that the sheaf $\mathcal{K}_X$ is flasque, i.e. for any open subset $U \subseteq X$, the map $\mathcal{K}(X) \to \mathcal{K}(U)$ is surjective. An important consequence of this is *acyclicity* – all higher cohomology vanishes. Now, $\mathcal{K}_X^*$ is flasque (hence, acyclic) because $\mathcal{K}_X$, is, implying that $H^1(X, \mathcal{K}_X^*) = 0$!

We conclude that $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$, which is to say that line bundles (up to isomorphism) correspond to Cartier divisors (up to linear equivalence). Interpretations and consequences of this isomorphism to come.