Stable rationality and decomposition of the diagonal

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Introduction

A variety X/k of dimension n is *rational* if $X \dashrightarrow \mathbb{P}^n_k$ is birational, *stably rational* if $X \times \mathbb{P}^r$ is rational for some r, and *retract rational* if some dense open $U \subseteq X$ admits $U \xrightarrow{s} V \xrightarrow{r} U$ with $V \subseteq \mathbb{A}^m$ open and $rs = \mathrm{id}_U$. These get weaker left to right. The difficulty is that a Fano variety can have trivial π_1, no holomorphic forms, and small Hodge numbers, yet fail to be stably rational, so one needs an invariant surviving both birational modification and the addition of free $\mathbb{P}^r$-factors.

The decomposition of the diagonal supplies one, by moving the question onto cycles on $X \times X$. The diagonal Δ_X acts as the identity correspondence, so any rewriting of its class $[\Delta_X]$ transfers to every invariant on which correspondences act. The theorem at the center is that

X is universally CH_0-trivial if and only if $[\Delta_X] = [X \times z] + Z$ with z a degree-one zero-cycle and Z supported on $D \times X$ for a proper closed $D \subsetneq X$,

and stable rationality forces the left side. The method becomes effective through specialization: degenerate X to a singular Y whose resolution $\widetilde{Y}$ carries a computable obstruction, a Brauer class, an unramified class, or a differential form; if the general fiber were stably rational its decomposition would specialize to $\widetilde{Y}$ and kill the obstruction.

Throughout, X is smooth projective geometrically integral of dimension n over k, and $K = k(X)$.

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1. Zero-cycles and the first implication

Zero-cycles. $Z_0(X) = \bigoplus_{x \text{ closed}} \mathbb{Z}\,[x]$; rational equivalence is generated by $\mathrm{div}_C(f)$ for integral proper curves $C \subseteq X$ and $f \in k(C)^\times$; the quotient is $\mathrm{CH}_0(X)$. A principal divisor on a proper curve has degree zero, so

$\deg\big(\sum n_i[x_i]\big) = \sum n_i[k(x_i):k]$ descends to $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$. Over a nonclosed field a degree-one cycle need not be one rational point, only an integral combination of closed points with residue degrees of gcd 1.

X is CH_0-**trivial** if $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism, and **universally** CH_0-trivial if X_L is CH_0-trivial for every field L/k. Concretely: over every L there is a degree-one cycle, and every degree-zero cycle is rationally trivial.

Projective space. The model computation.

For every L, $\deg : \mathrm{CH}_0(\mathbb{P}^n_L) \xrightarrow{\sim} \mathbb{Z}$.

Proof. For $n = 1$: a degree-d closed point $x = \{F = 0\} \subset \mathbb{P}^1_L$ with F irreducible of degree d, and $o = \{t = 0\}$, gives $\mathrm{div}(F/t^d) = [x] - d[o]$, so $[x] = d[o]$ and $\alpha = \deg(\alpha)[o]$ for all α. For general n, the projective bundle formula $\mathrm{CH}_i(\mathbb{P}^n_L) \cong \bigoplus_{j=0}^{n} \mathrm{CH}_{i-j}(\operatorname{Spec} L)$, a free module on $1, \xi, \ldots, \xi^n$, at $i = 0$ leaves only the $j = 0$ term $\mathrm{CH}_0(\operatorname{Spec} L) = \mathbb{Z}$. □

The same formula gives the stability facts. For a rank-r bundle E on proper X, $\mathrm{CH}_i(\mathbb{P}(E)) \cong \bigoplus_{j=0}^{r-1} \mathrm{CH}_{i-j}(X)$, so at $i = 0$ only $j = 0$ survives and

$$\pi_* : \mathrm{CH}_0(\mathbb{P}(E)) \xrightarrow{\sim} \mathrm{CH}_0(X), \qquad \text{compatibly with base change.}$$

Birational invariance. For smooth proper X, Y in characteristic 0, birational implies $\mathrm{CH}_0(X_L) \cong \mathrm{CH}_0(Y_L)$ for all L. By weak factorization it suffices to compare X with a blow-up $\pi : \widetilde{X} = \mathrm{Bl}_Z X \to X$ along smooth Z of codimension $c \geq 2$; the blow-up formula

$$\mathrm{CH}_i(\widetilde{X}) \cong \mathrm{CH}_i(X) \oplus \bigoplus_{j=1}^{c-1} \mathrm{CH}_{i-j}(Z)$$

at $i = 0$ kills every correction ($\mathrm{CH}_{-j}(Z) = 0$), giving $\pi_* : \mathrm{CH}_0(\widetilde{X}) \xrightarrow{\sim} \mathrm{CH}_0(X)$.

A smooth proper stably rational variety over a field of characteristic 0 is universally CH_0-trivial.

Proof. For any L, with $X \times \mathbb{P}^r$ birational to $\mathbb{P}^{n+r}$,

$$\mathrm{CH}_0(X_L) \xleftarrow[\sim]{\pi_*} \mathrm{CH}_0(X_L \times \mathbb{P}^r_L) \xrightarrow{\sim} \mathrm{CH}_0(\mathbb{P}^{n+r}_L) \xrightarrow[\sim]{\deg} \mathbb{Z}$$

uses the bundle isomorphism, birational invariance, and the model computation; the composite is $\deg$. □

2. The diagonal as a correspondence

Since X is smooth, $X \times X$ is smooth of dimension $2n$ and carries the full intersection product. Let $p_1, p_2 : X \times X \to X$ be the projections. A class $\Gamma \in \mathrm{CH}_n(X \times X) = \mathrm{CH}^n(X \times X)$ acts on zero-cycles by

$$\Gamma_*(\beta) = (p_2)_*\big(p_1^*\beta \cdot \Gamma\big).$$

Degrees track through: $\beta \in \mathrm{CH}^n(X)$, so $p_1^*\beta \in \mathrm{CH}^n(X \times X)$, the product with $\Gamma \in \mathrm{CH}^n$ lands in $\mathrm{CH}^{2n}(X \times X) = \mathrm{CH}_0(X \times X)$, and $(p_2)_*$ returns to $\mathrm{CH}_0(X)$. Two computations are all we need.

For $\beta \in \mathrm{CH}_0(X)$ and $z \in \mathrm{CH}_0(X)$,

$$[\Delta_X]_*(\beta) = \beta, \qquad [X \times z]_*(\beta) = \deg(\beta)\, z.$$

Proof. Let $\delta : X \hookrightarrow X \times X$, so $[\Delta_X] = \delta_*[X]$ and $p_1\delta = p_2\delta = \mathrm{id}$. By the projection formula for δ,

$$p_1^*\beta \cdot \delta_*[X] = \delta_*\big(\delta^* p_1^*\beta\big) = \delta_*\beta,$$

and applying $(p_2)_*$ gives $[\Delta_X]_*\beta = (p_2\delta)_*\beta = \beta$.

For the product cycle, $[X \times z] = p_2^* z$, so by the projection formula for p_2,

$$[X \times z]_*\beta = (p_2)_*\big(p_1^*\beta \cdot p_2^* z\big) = (p_2)_*(p_1^*\beta) \cdot z.$$

For $\beta = [x]$ of degree d, the cycle $p_1^{-1}(x) = x \times X$ maps isomorphically to X under p_2 (after the base change $k \to k(x)$), so $(p_2)_*(p_1^*[x]) = d\,[X]$; by linearity $(p_2)_*(p_1^*\beta) = \deg(\beta)[X]$, and the claim follows. □

X admits an **integral decomposition of the diagonal** if

$$[\Delta_X] = [X \times z] + Z \quad \text{in } \mathrm{CH}_n(X \times X),$$

for a degree-one $z \in \mathrm{CH}_0(X)$, a proper closed $D \subsetneq X$, and Z with $\mathrm{Supp}(Z) \subseteq D \times X$.

Integral coefficients matter: a decomposition in $\mathrm{CH}_n \otimes \mathbb{Q}$ exists for every rationally connected variety (Bloch–Srinivas), so the obstructions below all live in torsion. The cohomological version, with $[\Delta_X] = [X \times z] + Z$ imposed only in $H^{2n}(X \times X, \mathbb{Z})$, is implied by the Chow one and is where torsion obstructions are read off.

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3. The decomposition criterion

The generic point is a K-point $\eta_X \in X(K)$, giving $[\eta_X] \in \mathrm{CH}_0(X_K)$ of degree 1.

Criterion (Bloch–Srinivas, Colliot-Thélène–Pirutka). The following are equivalent:

1. X is universally CH_0-trivial;
2. $[\eta_X] = z_K$ in $\mathrm{CH}_0(X_K)$ for some degree-one $z \in \mathrm{CH}_0(X)$;
3. X admits an integral decomposition of the diagonal.

(1) $\Rightarrow$ (2). Over K, $\deg : \mathrm{CH}_0(X_K) \xrightarrow{\sim} \mathbb{Z}$; both $[\eta_X]$ and z_K have degree 1, so they agree.

(2) $\Rightarrow$ (3): spreading out. The engine is a continuity property.

Restriction induces $\mathrm{CH}_0(X_K) \cong \varinjlim_{\varnothing \neq U \subseteq X} \mathrm{CH}_n(U \times X)$, the colimit over dense opens U of the first factor.

Proof. The generic fiber of p_1 is $X_K = \varprojlim_U (U \times X)$, with flat affine transition maps, and Chow groups send such filtered inverse limits of schemes to filtered direct limits of groups. Explicitly: a closed point of X_K is an integral subscheme of some $U \times X$ finite over U, hence a dimension-n cycle there (surjectivity); a rational equivalence over K comes from a function on a K-curve, and curve plus function are defined over the ring of some U, so the relation already holds in $\mathrm{CH}_n(U \times X)$ after shrinking (injectivity). □

Now restrict Δ_X and $X \times z$ to the generic fiber of p_1: they become $[\eta_X]$ and z_K, equal by (2). So $[\Delta_X] - [X \times z]$ dies in $\mathrm{CH}_0(X_K)$, hence in $\mathrm{CH}_n(U \times X)$ for some dense open U by the lemma. Put $D = X \setminus U$. The localization sequence

$$\mathrm{CH}_n(D \times X) \xrightarrow{\iota_*} \mathrm{CH}_n(X \times X) \xrightarrow{j^*} \mathrm{CH}_n(U \times X) \to 0$$

shows $[\Delta_X] - [X \times z] = \iota_* Z$ for some Z supported on $D \times X$. That is the decomposition.

(3) $\Rightarrow$ (1). Fix L/k and base-change: $[\Delta_{X_L}] = [X_L \times z_L] + Z_L$, $\mathrm{Supp}(Z_L) \subseteq D_L \times X_L$. Take $\beta \in \mathrm{CH}_0(X_L)$. The moving lemma (valid since X_L is smooth) replaces β by a rationally equivalent cycle with support disjoint from D_L. Apply the correspondence identity to β:

$$\underbrace{[\Delta_{X_L}]_*\beta}_{=\beta} = \underbrace{[X_L \times z_L]_*\beta}_{=\deg(\beta)\, z_L} + \underbrace{Z_{L*}\beta}_{=0},$$

where the last term vanishes because $p_1^*\beta \cdot Z_L$ is supported on $(\mathrm{Supp}\,\beta \cap D_L) \times X_L = \varnothing$. So $\beta = \deg(\beta)\, z_L$: every cycle is a degree multiple of z_L, and $\deg$ is an isomorphism. As L was arbitrary, (1) holds. ■

Together with Section 1,

X stably rational $\Rightarrow$ X universally CH_0-trivial $\Leftrightarrow$ X decomposes the diagor

The same generic-point argument upgrades to retract rationality: if $f : X \dashrightarrow \mathbb{P}^m$, $g : \mathbb{P}^m \dashrightarrow X$ have $gf = \mathrm{id}$, then in $\mathrm{CH}_0(\mathbb{P}^m_K) = \mathbb{Z}$ we have $[f(\eta_X)] = [P_0]$ for a constant point P_0; applying the graph correspondence Γ_g gives $[\eta_X] = \Gamma_{g*}[f(\eta_X)] = \Gamma_{g*}[P_0] = z_K$, which is (2). So retract rational varieties decompose the diagonal too. The converse fails: some varieties of general type decompose the diagonal, so a failed decomposition proves non-stable-rationality but never rationality.

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4. Consequences of a decomposition

Apply $[\Delta_X] = [X \times z] + Z$ to any invariant with a correspondence action. The identity becomes constant plus a term through $\dim D < n$; on invariants of coniveau zero the last term drops and the invariant collapses to a point's.

4.1 The Brauer group

For smooth integral X, restriction $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$ is injective (Auslander–Goldman). Evaluate $\alpha \in \mathrm{Br}(X)$ on a zero-cycle $\xi = \sum n_i[x_i]$ by

$$\alpha(\xi) = \sum_i n_i \,\mathrm{Cor}_{k(x_i)/k}\big(\alpha(x_i)\big) \in \mathrm{Br}(k), \qquad \alpha(x_i) := \alpha|_{k(x_i)}.$$

This is well defined on CH_0: for a proper curve $C \subseteq X$ and $f \in k(C)^\times$, the reciprocity law $\sum_{x \in C} \mathrm{Cor}_{k(x)/k}\, \partial_x(\alpha|_{k(C)}) = 0$ makes $\alpha(\mathrm{div}_C\, f) = 0$.

If X is smooth proper geometrically integral and universally CH_0-trivial, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}(X)$.

Proof. Fix a degree-one $z = \sum n_i[x_i]$, so $\sum n_i[k(x_i) : k] = 1$.

Injectivity. If $b \in \mathrm{Br}(k)$ dies on X, then $b|_{k(x_i)} = 0$, so $b = \big(\sum n_i[k(x_i) : k]\big)b = b(z) = 0$ by $\mathrm{Cor} \circ \mathrm{Res} = [k(x_i) : k]$.

Surjectivity. Given $\alpha \in \mathrm{Br}(X)$, set $\alpha' = \alpha - \pi^*\alpha(z)$, so $\alpha'(z) = 0$ and it suffices to show $\alpha' = 0$. Over $K = k(X)$ the criterion gives $[\eta_X] = z_K$, and evaluation respects rational equivalence, so $\alpha'(\eta_X) = \alpha'(z_K) = (\alpha'(z))_K = 0$. But $\alpha'(\eta_X)$ is the image of α' under $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K)$, which is injective, so $\alpha' = 0$. □

Over $\overline{k}$, $\mathrm{Br}(\overline{k}) = 0$, so a nonzero Brauer class obstructs universal triviality, hence stable rationality. This is the Artin–Mumford invariant.

4.2 Unramified cohomology

Fix m invertible in k. For $x \in X^{(1)}$ there are residues $\partial_x : H^i(k(X), \mu_m^{\otimes j}) \to H^{i-1}(k(x), \mu_m^{\otimes (j-1)})$, and

$$H^i_{\mathrm{nr}}(k(X)/k, \mu_m^{\otimes j}) = \bigcap_{x \in X^{(1)}} \ker \partial_x.$$

For $(i,j) = (2,1)$ this is $\mathrm{Br}(X)[m]$ (Merkurjev–Suslin, via the Bloch–Ogus complex, the input we cite). These groups are stable-birational invariants, and the evaluation pairing on zero-cycles works exactly as for Br: an unramified class specializes to points and is killed on rational equivalence by residue reciprocity on curves. So the proof of 4.1 gives, verbatim, that universal triviality makes $H^i(k, \mu_m^{\otimes j}) \to H^i_{\mathrm{nr}}(k(X)/k, \mu_m^{\otimes j})$ an isomorphism. Any nonzero normalized class in H^i_{nr} (for any i) is an obstruction, extending the reach past Br.

4.3 Differential forms

This is the obstruction that operates in characteristic p. Correspondences act on the coherent cohomology $\bigoplus_{p,q} H^q(X, \Omega^p_X)$ over any field: cycle classes lie in $H^p(X, \Omega^p)$, proper maps have Gysin pushforwards shifting bidegree by the codimension, and

$$\Gamma_*(\alpha) = (p_2)_*\big(p_1^*\alpha \cup \mathrm{cl}(\Gamma)\big)$$

preserves the bidegree (p,q), so it acts on global forms $H^0(X, \Omega^i) = H^{i,0}(X)$. Two support lemmas control it.

If $\mathrm{Supp}(\Gamma) \subseteq T \times X$ with $t = \dim T$, then Γ_* on $H^{i,0}(X)$ factors through restriction $\mu^* : H^{i,0}(X) \to H^{i,0}(\widetilde{T})$ for a resolution $\mu : \widetilde{T} \to T \hookrightarrow X$. In particular $\Gamma_* = 0$ when $i > t$.

Proof. Write $\mathrm{cl}(\Gamma) = (\mu \times \mathrm{id})_*\gamma$. The projection formula gives

$$\Gamma_*\alpha = (p_2)_*(\mu \times \mathrm{id})_*\big(q_1^*(\mu^*\alpha) \cup \gamma\big) = (q_2)_*\big(q_1^*(\mu^*\alpha) \cup \gamma\big),$$

with q_1, q_2 the projections of $\widetilde{T} \times X$, so $\Gamma_*\alpha$ depends only on $\mu^*\alpha \in H^{i,0}(\widetilde{T})$. If $i > t = \dim \widetilde{T}$ then $\Omega^i_{\widetilde{T}} = 0$, so $\mu^*\alpha = 0$. □

If $\mathrm{Supp}(\Gamma) \subseteq X \times T$ with $\mathrm{codim}_X T \geq 1$, then $\Gamma_* = 0$ on $H^{i,0}(X)$ for every i.

Proof. Write $\mathrm{cl}(\Gamma) = (\mathrm{id} \times \iota)_*\delta$ for $\iota : \widetilde{T} \to T \hookrightarrow X$ of codimension $c \geq 1$. As above,

$$\Gamma_*\alpha = (\iota)_*\,(r_2)_*\big(r_1^*\alpha \cup \delta\big) \in \mathrm{Image}\big(\iota_* : H^\bullet(\widetilde{T}) \to H^\bullet(X)\big).$$

The Gysin map ι_* raises bidegree by (c,c), so it maps into $\bigoplus_{b\geq c} H^\bullet(X,\Omega^\bullet)$ with second index $\geq c \geq 1$; it never hits $H^{i,0}$, which has second index 0. Hence $\Gamma_*\alpha = 0$. □

A decomposition kills all forms (Bloch–Srinivas, Totaro). If X is smooth proper over k and admits a decomposition of the diagonal, then $H^0(X,\Omega^i_X) = 0$ for all $i \geq 1$, in any characteristic.

Proof. Take the **transpose** of $[\Delta_X] = [X \times z] + Z$ by swapping the two factors. Transposition fixes the symmetric diagonal, sends $X \times z$ to $z \times X$, and sends Z (supported on $D \times X$) to Z' (supported on $X \times D$):

$$[\Delta_X] = [z \times X] + Z', \qquad \operatorname{Supp}(Z') \subseteq X \times D.$$

Act on $\alpha \in H^{i,0}(X)$ with $i \geq 1$:

$$\alpha = [\Delta_X]_*\alpha = [z \times X]_*\alpha + Z'_*\alpha.$$

Now $z \times X$ has first-factor support $\operatorname{Supp}(z) \times X$ with $\dim \operatorname{Supp}(z) = 0 < i$, so Lemma "first factor" gives $[z \times X]_*\alpha = 0$; and Z' has second-factor support in $X \times D$ with $\operatorname{codim} D \geq 1$, so Lemma "second factor" gives $Z'_*\alpha = 0$. Therefore $\alpha = 0$. □

The transpose is what makes the argument work in every degree at once: after transposing, the term with the large ($\dim D = n-1$) support sits in the second factor, where any positive codimension already kills forms, while the small-support term sits in the first factor, where dimension zero kills forms of positive degree. Over $\mathbb{C}$ this says nothing new for Fano varieties, but in characteristic p an inseparable construction can force $H^0(\widetilde{Y},\Omega^i) \neq 0$ on a variety lifting to a complex Fano, and then no decomposition can exist (Section 6.4).

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5. Specialization

Let R be a DVR with fraction field K, residue field k, and uniformizer π_R, and let $\mathcal{X} \to \operatorname{Spec} R$ be flat proper with generic fiber $X = \mathcal{X}_K$ and special fiber $Y = \mathcal{X}_k$ (possibly singular), with resolution $\tau : \widetilde{Y} \to Y$.

Specialization of cycles. Fulton's specialization homomorphism $\mathrm{sp} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y)$ is the composite

$$\mathrm{CH}_i(X) \xleftarrow[\text{onto}]{} \mathrm{CH}_i(\mathcal{X}) \xrightarrow{\cap\,[Y]} \mathrm{CH}_i(Y),$$

where the left map is restriction to the generic fiber (surjective, with a chosen splitting) and the right is the Gysin map for the principal divisor $Y = \operatorname{div}(\pi_R)$. Independence of the splitting is the content of the theorem. It respects proper

pushforward, flat pullback, and products, sends flat-over-R subvarieties to their special fibers, and so sends $[\Delta_X] \mapsto [\Delta_Y]$ and degree-one cycles to degree-one cycles.

Universally CH_0-trivial morphisms. Call $\tau : \widetilde{Y} \to Y$ universally CH_0-trivial if $\tau_* : \mathrm{CH}_0(\widetilde{Y}_L) \xrightarrow{\sim} \mathrm{CH}_0(Y_L)$ for all L. The usable criterion:

If every fiber $\widetilde{Y}_y$ (over $k(y)$, for every scheme point $y \in Y$) is universally CH_0-trivial, then τ is universally CH_0-trivial.

Idea. Surjectivity of τ_* is clear. For injectivity, filter Y by the dimension of the locus with uniform fiber behavior and induct on $\dim Y$ using the localization sequence: over each stratum's generic point the fiber is universally trivial, which lifts and matches zero-cycles; the localization sequences glue these into a global isomorphism. (Full argument in Colliot-Thélène–Pirutka.) □

In practice the fibers are a point over the smooth locus and, over singular points, a rational variety with a rational point, most often the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ from blowing up an ordinary double point, or a tree of $\mathbb{P}^k$'s. Each is universally CH_0-trivial.

Specialization of the decomposition. Suppose $X_{\overline{K}}$ is smooth, Y is geometrically integral, and $\tau : \widetilde{Y} \to Y$ is a universally CH_0-trivial resolution with $\widetilde{Y}$ smooth proper. If $X_{\overline{K}}$ is universally CH_0-trivial, so is $\widetilde{Y}$. Contrapositively, an obstruction on $\widetilde{Y}$ implies $X_{\overline{K}}$ is not stably rational.

Proof, in three acts. *Stage 1 (descend).* A decomposition over $\overline{K}$ uses finitely many cycles and rational equivalences, so it is defined over a finite extension K'/K. Replace R by (a localization of) its integral closure in K'; universal triviality descends through the finite residue extension by corestriction, so we may assume the decomposition is defined over X itself:

$$[\Delta_X] = [X \times z] + Z, \quad \mathrm{Supp}(Z) \subseteq D \times X.$$

Stage 2 (specialize). Apply sp on $\mathcal{X} \times_R \mathcal{X}$. Being a ring homomorphism compatible with the correspondence operations, it sends the decomposition to

$$[\Delta_Y] = [Y \times z_0] + Z_0, \quad \mathrm{Supp}(Z_0) \subseteq D_0 \times Y,$$

with $z_0 = \mathrm{sp}(z)$ of degree one. This is a decomposition for the singular Y.

Stage 3 (cross τ). Let $F = k(\widetilde{Y})$; the generic point of $\widetilde{Y}$ maps to $\eta \in \mathrm{CH}_0(Y_F)$ of degree one. Running the (3)⇒(1) argument on the specialized decomposition, valid because the generic point lies in the smooth locus of Y where the moving

lemma applies, gives $\eta = (z_0)_F$ in $\mathrm{CH}_0(Y_F)$. Since $\tau_* : \mathrm{CH}_0(\widetilde{Y}_F) \xrightarrow{\sim} \mathrm{CH}_0(Y_F)$, applying τ_*^{-1} yields $[\eta_{\widetilde{Y}}] = \widetilde{z}_F$ for a fixed degree-one $\widetilde{z}$. That is condition (2) for $\widetilde{Y}$, so $\widetilde{Y}$ decomposes the diagonal. □

Very general members. For a projective family $\mathcal{X} \to B$ over $\mathbb{C}$, the fibers admitting a decomposition form a countable union of closed subvarieties of B: a decomposition is witnessed by bounded-degree cycles on $\mathcal{X}_b \times \mathcal{X}_b$ and bounded-degree rational equivalences on $\mathcal{X}_b \times \mathcal{X}_b \times \mathbb{P}^1$, parametrized by finitely many relative Hilbert schemes (proper over B) in each of countably many numerical types; the decomposition condition is closed, and properness makes each image closed. So if one fiber has no decomposition, the very general fiber has none. This is the closing step of every application: one degeneration plus countability removes a decomposition from the very general member.

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6. Four applications

For a discretely valued field (F, v) with residue field κ and $\mathrm{char} \neq 2$, the residue of a quaternion class is

$$\partial_v(f, g) = (-1)^{v(f)v(g)} \overline{\left(\frac{f^{\,v(g)}}{g^{\,v(f)}}\right)} \;\in\; \kappa^\times/(\kappa^\times)^2 = H^1(\kappa, \mathbb{Z}/2),$$

and for a smooth surface S the Bloch–Ogus complex identifies the unramified group as the kernel of the total residue:

$$H^2_{\mathrm{nr}}(\mathbb{C}(S)/\mathbb{C}, \mathbb{Z}/2) = \ker\Big({}_2\mathrm{Br}(\mathbb{C}(S)) \xrightarrow{\oplus_C \partial_C} \bigoplus_{C \in S^{(1)}} H^1(\mathbb{C}(C), \mathbb{Z}/2)\Big).$$

These two formulas drive the first three examples.

6.1 Artin–Mumford: torsion in Br

Let $M(x)$ be a general symmetric 4×4 matrix of linear forms on $\mathbb{P}^3$. The symmetroid $S = \{\det M = 0\}$ is a quartic with a node wherever $\mathrm{rk}\, M \leq 2$, which is 10 points for general M. The double solid $W = \{w^2 = \det M\} \to \mathbb{P}^3$ is singular over the nodes, and $\widetilde{V}$ is its resolution.

The rank-two degeneration gives $\widetilde{V}$ a conic bundle model $\pi : V \to \mathbb{P}^2$: the generic fiber is the conic attached to a quaternion algebra $(a, b) \in {}_2\mathrm{Br}(F)$, $F = \mathbb{C}(\mathbb{P}^2)$, and it degenerates to a line-pair over the discriminant $\Delta = \Delta_1 \cup \cdots \cup \Delta_r \subset \mathbb{P}^2$. Over each Δ_i the two lines are swapped by an étale double cover $\varepsilon_i : \widetilde{\Delta}_i \to \Delta_i$, whose class is exactly the residue

$$\partial_{\Delta_i}(a, b) = [\varepsilon_i] \in H^1(\mathbb{C}(\Delta_i), \mathbb{Z}/2).$$

The class (a, b) itself is ramified, so it is not the obstruction; the obstruction is the unramified class assembled from the covers. Concretely $\mathrm{Br}(\widetilde{V})$ is computed from the double-cover data by the residue sequence above together with the second residues ∂_P at the points $P \in \Delta_i \cap \Delta_j$: a collection (ε_i) defines an unramified class precisely when ∂_P of the residues cancels at every P (compatibility along the two branches), and that class is nonzero precisely when the ε_i do not come from a single global square class.

For the symmetroid configuration Artin and Mumford verify both conditions, so

$$\mathrm{Br}(\widetilde{V}) \cong \mathbb{Z}/2 \neq 0, \qquad \text{equivalently } H^3(\widetilde{V}, \mathbb{Z})_{\mathrm{tors}} \cong \mathbb{Z}/2$$

(the second because $\widetilde{V}$ is rationally connected, so $H^2(\mathcal{O}) = 0$ and the Brauer group is all torsion). By Proposition 4.1, $\mathrm{Br}(\widetilde{V}) \neq 0$ rules out universal CH_0-triviality: $\widetilde{V}$ is unirational but not stably rational. The nonvanishing computation for the specific configuration is the one fact cited to Artin–Mumford.

6.2 Voisin: quartic double solids

A smooth quartic double solid (a double cover of $\mathbb{P}^3$ branched on a smooth quartic) is Fano and unirational, so its rational decomposition exists and only the integral one can obstruct. Take a family $\{Y_t\}$ of quartic double solids degenerating to Y_0, whose branch quartic acquires the symmetroid's 10 nodes; Y_0 is then an Artin–Mumford double solid with 10 nodes.

Resolution. An ordinary double point of a threefold is, analytically, the affine cone $\{xy = zw\} \subset \mathbb{A}^4$ over the smooth quadric surface $Q = \{xy = zw\} \subset \mathbb{P}^3 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Blowing up the point replaces it by the projectivized tangent cone, which is exactly Q, and the total space becomes smooth. So $\tau : \widetilde{Y}_0 \to Y_0$ has every fiber a point (over the smooth locus) or a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ (over each node). Both are universally CH_0-trivial (rational with a rational point), so by the fiberwise criterion τ is universally CH_0-trivial.

Obstruction and propagation. $\widetilde{Y}_0$ is an Artin–Mumford threefold, hence $\mathrm{Br}(\widetilde{Y}_0) = \mathbb{Z}/2 \neq 0$ and it has no decomposition. By the specialization theorem the geometric generic fiber has no decomposition, and by countability the very general quartic double solid is not stably rational.

The general smooth Y_t has torsion-free cohomology, so the $\mathbb{Z}/2 \subset H^3$ that obstructs Y_0 is created by the degeneration and invisible on Y_t. What survives specialization is the failure of the diagonal to decompose, which is strictly finer than any cohomological invariant of the smooth fiber.

6.3 Colliot-Thélène–Pirutka: quartic threefolds

Smooth quartic threefolds $X_4 \subset \mathbb{P}^4$ are nonrational (Iskovskikh–Manin, birational rigidity), but that says nothing stable. The stable result comes from a quadric-bundle degeneration.

The special quartic $Y \subset \mathbb{P}^4$ is chosen with an equation quadratic in part of the variables, so that a projection $Y \dashrightarrow \mathbb{P}^2$ is, after blow-up, a quadric surface bundle: over $F = \mathbb{C}(\mathbb{P}^2)$ the generic fiber is $\{q = 0\} \subset \mathbb{P}^3_F$ for a rank-four form $q = \langle a_1, a_2, a_3, a_4 \rangle$, $a_i \in F^\times$. Its invariants are

$$\operatorname{disc}(q) = a_1a_2a_3a_4 \in F^\times/(F^\times)^2, \qquad c(q) = \sum_{i<j} (a_i, a_j) \in {}_2\mathrm{Br}(F),$$

the discriminant and the Clifford (Hasse–Witt) invariant; equivalently $c(q)$ is the class of the even Clifford algebra $C_0(q)$, a quaternion algebra over the étale quadratic extension $F[\sqrt{\operatorname{disc} q}]$. The quadric has an F-point iff q is isotropic, so $c(q)$ is the Brauer obstruction of the bundle. Its residue along a component Δ_i of the degeneration curve $\Delta = \{\det q = 0\} = \{a_1a_2a_3a_4 = 0\}$ is again a double cover of Δ_i (the two rulings of the corank-one degenerate quadric), computed by the same ∂_{Δ_i} formula.

Colliot-Thélène and Pirutka choose the a_i so that

$$c(q) \in H^2_{\mathrm{nr}}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) = \mathrm{Br}(\widetilde{Y})[2] \quad \text{is nonzero:}$$

the residues along the Δ_i are compatible at every intersection point (unramified), and the double-cover data is not globally trivial (nonzero). This residue computation, of Artin–Mumford type, is the first cited fact.

The technical crux, harder than for nodal double solids, is a universally CH_0-trivial resolution $\tau : \widetilde{Y} \to Y$. The singular locus of Y is where the quadric bundle degenerates badly (the corank jumps or Δ is singular); Colliot-Thélène and Pirutka resolve it by explicit blow-ups whose fibers are, over each stratum, a quadric or a rational surface with a rational point, and over finitely many worse points a tree of rational varieties, all universally CH_0-trivial. The fiberwise criterion then gives τ universally CH_0-trivial. This case analysis is the second cited fact.

With $\mathrm{Br}(\widetilde{Y}) \neq 0$ and τ universally CH_0-trivial, specialization plus countability give: the very general quartic threefold is not stably rational.

6.4 Totaro: hypersurfaces and forms in characteristic p

Totaro's obstruction is a differential form produced by inseparable geometry, and it reaches all dimensions. Over $\mathbb{C}$ a Fano hypersurface has no forms, but characteristic p is different: a degree-p cover

$$Y : \{y^p = f(x_0, \dots, x_n)\} \to \mathbb{P}^n$$

is inseparable ($d(y^p) = 0$), and its resolution acquires global forms unavailable in characteristic 0. Concretely, for the analogous inseparable double cover in characteristic 2, branched over $B = \{f = 0\}$, the relation $w^2 = f$ gives $0 = d(w^2) = f'$, and the closed form df on $\mathbb{P}^n$, together with the ramification along B, produces a nonzero section of Ω^{n-1} on a resolution. Kollár's theorem makes this precise: for f general of the right degree, a resolution $\widetilde{Y}$ has

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) \neq 0,$$

the existence of the form and the general-position hypothesis being what we cite. The exceptional fibers of the resolution are rational with rational points, so τ is universally CH_0-trivial.

By Theorem 4.3, a decomposition of the diagonal would force $H^0(\widetilde{Y}, \Omega^{n-1}) = 0$, so $\widetilde{Y}$ has none. To reach a complex hypersurface, Totaro spreads $X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ over a DVR of mixed characteristic and degenerates to Kollár's Y_0 over $\overline{\mathbb{F}}_p$; a decomposition on the general complex fiber would specialize (Section 5.3, with the universally CH_0-trivial resolution of Y_0) to $\widetilde{Y}_0$ and kill the form, a contradiction. Countability then gives the very general statement, with the degree range in which a suitable inseparable model and resolution exist:

$$d \geq 2\left\lceil \tfrac{n+2}{3} \right\rceil \implies \text{a very general hypersurface of dimension } n,\ \deg d \text{ is not st}$$

For $n = 3$ this is $d \geq 4$ (recovering quartic threefolds by a different obstruction), for $n = 4$ it is $d \geq 4$ (very general quartic fourfolds), and it continues in every dimension. The two mechanisms of Section 6 thus contrast cleanly: Voisin and Colliot-Thélène–Pirutka obstruct by torsion in Br or H^i_{nr}, Totaro by an inseparable differential form invisible to all of them.

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