Reciprocity: Weil, Hilbert, and Artin

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Introduction

Three theorems, unrelated at first glance, are actually cut from the same cloth.

The residue theorem, from the theory of Riemann surfaces, says that a meromorphic differential ω on a compact Riemann surface has residues summing to zero:

$$\sum_p \operatorname{Res}_p(\omega) = 0.$$

The degree formula, from algebraic geometry, says that a rational function f on a smooth projective curve has local orders of vanishing summing to zero:

$$\sum_p v_p(f)\,[k(p):k] = 0.$$

The product formula, from number theory, says that a nonzero element a of a number field has local absolute values multiplying to one:

$$\prod_v |a|_v = 1.$$

Each statement attaches a local invariant at every point or place of a global object, and then asserts that the total is trivial. A function that is defined everywhere cannot have a net residue, a net order of vanishing, or a net size. The local contributions must somehow cancel, precisely because the object is global.

Reciprocity, in some sense, is the systematic study of this cancellation. This post follows one thread through it. We start on curves, where the divisor of a function has degree zero, refine that statement to a pairing of two functions called the tame symbol, and prove Weil reciprocity: the product of all local tame symbols is one. We then recognize the tame symbol as a boundary map in K-theory, which places Weil reciprocity one rung up a ladder whose bottom rung is the degree formula. Passing to number fields, the same ladder produces the product formula, Hilbert reciprocity, and finally Artin reciprocity, the complete abelian reciprocity law of class field theory. Along the way, we recover the classical quadratic reciprocity law of Gauss as a special case.

The key thrust is that *a global object has trivial total local boundary*. This idea recurs across adeles, Tate’s thesis, class field theory, the Brauer–Manin obstruction, higher local fields, and the analogy between curves and number fields.

The function field analogy

Borges famously said that “the original is unfaithful to the translation”. Indeed, curves and function fields may be conceptualized in the same breath. A smooth projective curve X over a field k has a function field $k(X)$, and each closed point $x \in X$ gives a valuation $v_x : k(X)^\times \to \mathbb{Z}$ recording the order of vanishing there. A number field K has, in place of points, its places: the nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ together with the archimedean embeddings. An element $a \in K^\times$ has a local size at every place, and the product formula constrains the total.

Geometric curve	**Arithmetic curve**
smooth projective curve X/k	$\operatorname{Spec} \mathcal{O}_K$ plus archimedean places
function field $k(X)$	number field K
closed point $x \in X$	place v of K
completed local field $k(X)_x$	local field K_v
valuation $v_x(f)$	valuation or absolute value at v
$\deg \operatorname{div}(f) = 0$	$\prod_v \lvert a\rvert_v = 1$

When $k = \mathbb{F}_q$ is a finite field, the analogy becomes literal. The field $\mathbb{F}_q(X)$ is a global field on exactly the same footing as a number field, and the two cases together are the content of the classical theory of global fields. When $k = \mathbb{C}$, the curve is a compact Riemann surface, and the objects are meromorphic functions and differentials. For a number field, the slogan “curve over $\mathbb{F}_1$” is best treated as a guiding metaphor: $\operatorname{Spec} \mathcal{O}_K$ behaves like a complete one-dimensional arithmetic object, with the archimedean places supplying the points at infinity that make it complete.

1. Valuations on curves and the divisor formula

Let C be a smooth projective geometrically integral curve over a field k, with function field $K = k(C)$. For each closed point $p \in C$, the local ring $\mathcal{O}_{C,p}$ is a discrete valuation ring; this is where smoothness enters, since a closed point on a smooth curve has codimension one and its local ring is regular of dimension one, hence a DVR.

Each p therefore determines a valuation $v_p : K^\times \to \mathbb{Z}$. Choosing a uniformizer t_p at p, every $f \in K^\times$ has a unique expression $f = t_p^{v_p(f)}\, u$ with $u \in \mathcal{O}_{C,p}^\times$, and $v_p(f)$ is the order of vanishing of f at p (negative if f has a pole there). The divisor of f collects this data over all points,

$$\operatorname{div}(f) = \sum_{p \in C} v_p(f)\,[p],$$

a finite sum because a rational function on a proper curve has only finitely many zeros and poles. The degree of a divisor $D = \sum_p n_p[p]$ is $\deg D = \sum_p n_p\,[k(p) : k]$, where the weight $[k(p) : k]$ is the residue degree, the number of geometric points the closed point p carries.

Degree of a principal divisor. For every $f \in k(C)^\times$, one has $\deg \operatorname{div}(f) = 0$.

Proof. If $f \in k^\times$ is a nonzero constant, then $v_p(f) = 0$ for every p, so $\operatorname{div}(f) = 0$.

Assume f is nonconstant. Then f defines a dominant morphism $f : C \to \mathbb{P}^1_k$, sending p to the value $f(p)$, and because C is projective and f is nonconstant this morphism is finite of some degree $d = [k(C) : k(t)]$, where t is the coordinate on $\mathbb{P}^1$. Finiteness means that every fiber has the same degree d when points are counted with ramification indices and residue degrees:

$$\sum_{p \mapsto q} e(p/q)\,[k(p) : k(q)] = d \qquad \text{for every closed } q \in \mathbb{P}^1.$$

Apply this at $q = 0$ and $q = \infty$. The zeros of f are exactly the points of the fiber over 0, with $v_p(f) = e(p/0)$, so the zero divisor $(f)_0 = \sum_{p \mapsto 0} v_p(f)[p]$ has degree $d\,[k(0) : k] = d$. The poles of f are the fiber over ∞, and the pole divisor $(f)_\infty = \sum_{p \mapsto \infty} -v_p(f)[p]$ likewise has degree $d\,[k(\infty) : k] = d$, since 0 and ∞ are k-rational. Therefore $\deg(f)_0 = \deg(f)_\infty$, and since $\operatorname{div}(f) = (f)_0 - (f)_\infty$ we conclude $\deg \operatorname{div}(f) = 0$. □

The prototype is $C = \mathbb{P}^1_k$ with coordinate t, where $\operatorname{div}(t - a) = [a] - [\infty]$: the function $t - a$ has a simple zero at a and a simple pole at infinity, and these cancel in degree. This is the geometric ancestor of the product formula for number fields.

There is a differential-forms companion to this statement, and it is worth naming because it is the residue theorem from the opening. The logarithmic derivative df/f is a meromorphic differential, and its residue at p is exactly the order of vanishing:

$$\operatorname{Res}_p\left(\frac{df}{f}\right) = v_p(f).$$

Over an algebraically closed field, where every residue degree is one, summing this identity over all p turns the residue theorem $\sum_p \operatorname{Res}_p(df/f) = 0$ into the degree formula $\sum_p v_p(f) = 0$. The order of vanishing is a residue. This is our clue into the fact that the degree formula and its refinements are residue theorems in disguise, which the tame symbol makes precise.

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2. The tame symbol

The divisor of a function records the local order of one function at a time. Weil reciprocity refines this to a pairing that takes two functions and produces, at each point, a residue-like invariant valued in the residue field.

Let $p \in C$ be a closed point with residue field $\kappa(p) = k(p)$. For $f, g \in K^\times$, the tame symbol at p is

$$(f,g)_p = (-1)^{\,v_p(f)\,v_p(g)} \left(\frac{f^{\,v_p(g)}}{g^{\,v_p(f)}}\right)(p) \;\in\; k(p)^\times.$$

The point of the exponents is that the rational function $f^{\,v_p(g)}/g^{\,v_p(f)}$ has valuation $v_p(g)v_p(f) - v_p(f)v_p(g) = 0$ at p. Having valuation zero, it is a unit in $\mathcal{O}_{C,p}$, so it can be evaluated modulo the maximal ideal to land in $k(p)^\times$. The evaluation is what the notation $(\,\cdot\,)(p)$ denotes.

To see the cancellation concretely, and to obtain the formula one actually computes with, write $a = v_p(f)$ and $b = v_p(g)$, choose a uniformizer $t = t_p$, and factor $f = t^a u$ and $g = t^b v$ with $u, v \in \mathcal{O}_{C,p}^\times$. Then

$$\frac{f^b}{g^a} = \frac{(t^a u)^b}{(t^b v)^a} = \frac{u^b}{v^a},$$

the powers of t having canceled, and therefore

$$(f,g)_p = (-1)^{ab}\,\frac{u(p)^b}{v(p)^a}.$$

This unit formula is the working definition. It also reveals what the tame symbol measures. The value $u(p)$ is the leading coefficient of f in the local expansion, the first nonzero coefficient after factoring out t^a, and likewise $v(p)$ is the leading coefficient of g. The tame symbol is a pairing of leading coefficients: it raises the leading coefficient of each function to the order of the other, takes the ratio, and corrects the sign. When both functions are units, both orders vanish and the symbol is trivially 1; the symbol only sees points where at least one function has a zero or a pole.

We record the algebraic properties that make the tame symbol behave like a residue pairing. Each is a short computation with the unit formula.

The tame symbol does not depend on the choice of uniformizer.

Proof. Replace t by another uniformizer $t' = wt$ with $w \in \mathcal{O}_{C,p}^{\times}$. Then $f = (t')^a(w^{-a}u)$ and $g = (t')^b(w^{-b}v)$, so the new unit parts are $w^{-a}u$ and $w^{-b}v$. Substituting into the unit formula,

$$(-1)^{ab}\frac{(w^{-a}u)(p)^b}{(w^{-b}v)(p)^a} = (-1)^{ab}\frac{w(p)^{-ab}\,u(p)^b}{w(p)^{-ab}\,v(p)^a} = (-1)^{ab}\frac{u(p)^b}{v(p)^a},$$

the factors $w(p)^{-ab}$ canceling. The value is unchanged. □

For fixed p, the tame symbol is multiplicative in each argument:

$$(ff', g)_p = (f,g)_p\,(f',g)_p, \qquad (f, gg')_p = (f,g)_p\,(f,g')_p.$$

Proof. Let $a = v_p(f)$, $a' = v_p(f')$, $b = v_p(g)$, and write $f = t^a u$, $f' = t^{a'}u'$, $g = t^b v$ with units u, u', v. Then $ff' = t^{a+a'}(uu')$, so

$$(ff', g)_p = (-1)^{(a+a')b}\frac{(uu')(p)^b}{v(p)^{a+a'}} = \left((-1)^{ab}\frac{u(p)^b}{v(p)^a}\right)\left((-1)^{a'b}\frac{u'(p)^b}{v(p)^{a'}}\right) = (f,g)_p\,($$

The argument in the second variable is identical. □

The tame symbol satisfies $(g,f)_p = (f,g)_p^{-1}$.

Proof. With $f = t^a u$ and $g = t^b v$,

$$(f,g)_p = (-1)^{ab}\frac{u(p)^b}{v(p)^a}, \qquad (g,f)_p = (-1)^{ab}\frac{v(p)^a}{u(p)^b},$$

and the product of the two is 1. The sign $(-1)^{ab}$ is exactly what makes the symbol antisymmetric rather than merely a ratio; without it the two expressions would not be inverse when ab is odd. □

The next relation is the deepest of the elementary properties, and is in fact the reason the tame symbol connects to K-theory. It says that the symbol never sees the pair $(f, 1-f)$.

If $f \in K^\times$ and $f \neq 1$, then $(f, 1-f)_p = 1$ for every $p \in C$.

Proof. Write $a = v_p(f)$ and split into cases according to the sign of a.

If $a > 0$, then f vanishes at p, so $1-f$ is a unit with $(1-f)(p) = 1$ and $v_p(1-f) = 0$. The unit formula gives $(f, 1-f)_p = (1-f)(p)^{-a} = 1^{-a} = 1$.

If $a = 0$, then f is a unit. Should $1 - f$ also be a unit, both valuations vanish and the symbol is 1. Should $1 - f$ vanish, with $b = v_p(1 - f) > 0$, the unit formula gives $(f, 1 - f)_p = f(p)^b$; but $1 - f$ vanishing at p means $f(p) = 1$, so again the symbol is 1.

If $a < 0$, then f has a pole at p, so f^{-1} vanishes and $1 - f^{-1}$ is a unit with value 1. From the identity

$$1 - f = -f\,(1 - f^{-1})$$

we read off $v_p(1 - f) = v_p(f) = a$. Writing $f = t^a u$ gives $1 - f = t^a v$ with unit part $v = -u(1 - f^{-1})$, hence $v(p) = -u(p)$. The unit formula then yields

$$(f, 1 - f)_p = (-1)^{a^2} \frac{u(p)^a}{v(p)^a} = (-1)^{a^2} \frac{u(p)^a}{(-u(p))^a} = (-1)^{a^2 - a}.$$

Since $a^2 - a = a(a - 1)$ is a product of consecutive integers, it is even, and the symbol is 1. $\square$

The Steinberg relation is the algebraic heart of what follows. It says the tame symbol annihilates the elements $\{f, 1 - f\}$ that define Milnor K-theory, and therefore descends to a boundary map out of K_2. We return to this in Section 4, once we have proved the global law.

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3. Weil reciprocity

Weil reciprocity is the global product formula for the tame symbol. It is to the tame symbol what the degree formula is to the valuation.

Weil reciprocity. Let C be a smooth projective geometrically integral curve over k, with function field $K = k(C)$. For $f, g \in K^\times$,

$$\prod_{p \in C} N_{k(p)/k}\,(f, g)_p = 1.$$

When k is algebraically closed, every residue field is k and the norms are trivial, so this reads $\prod_{p \in C} (f, g)_p = 1$.

The product is finite, since $(f, g)_p = 1$ at every point where both f and g are units, and this excludes all but the finitely many zeros and poles of f and g. We prove the algebraically closed case in full by reducing to $\mathbb{P}^1$, and then indicate the residue-field norm that handles a general base.

The proof on the projective line

Take $C = \mathbb{P}^1_k$ with coordinate t and k algebraically closed. The group $k(t)^\times$ is generated by the nonzero constants and the linear factors $t - a$ for $a \in k$, because a rational function factors into linear terms over an algebraically closed field. By bimultiplicativity it suffices to verify the product formula when f and g each run over this generating set.

First take $f = t - a$ and $g = t - b$ with $a \neq b$. The only points contributing are a, b, and ∞.

At $p = a$, one has $v_a(t-a) = 1$ and $v_a(t-b) = 0$, and since $t - b$ evaluates to $a - b$ there, the unit formula gives $(t-a, t-b)_a = (a-b)^{-1}$.

At $p = b$, symmetrically, $v_b(t-a) = 0$ and $v_b(t-b) = 1$, giving $(t-a, t-b)_b = b - a$.

At $p = \infty$, use the uniformizer $s = 1/t$. Then $t - a = s^{-1}(1 - as)$ and $t - b = s^{-1}(1 - bs)$, so both functions have valuation -1 and both unit parts $1 - as$, $1 - bs$ evaluate to 1 at $s = 0$. The unit formula gives $(t-a, t-b)_\infty = (-1)^{(-1)(-1)} = -1$.

Multiplying the three contributions,

$$(a-b)^{-1}\,(b-a)\,(-1) = \frac{b-a}{a-b}\,(-1) = (-1)(-1) = 1.$$

If instead $a = b$, then only a and ∞ contribute. At a the symbol $(t-a, t-a)_a$ equals -1 by antisymmetry, since a symbol equal to its own inverse and computed from the sign is $(-1)^{1\cdot 1} = -1$; at ∞ the same computation gives -1; and the product is 1.

Next take $f = c \in k^\times$ constant and $g = t - a$. At $p = a$, the unit formula gives $(c, t-a)_a = c^{v_a(t-a)} = c$ raised to the appropriate power, namely $(c, t-a)_a = c$. At $p = \infty$, where $v_\infty(t-a) = -1$, it gives $(c, t-a)_\infty = c^{-1}$. All other points contribute 1, and the product is $c \cdot c^{-1} = 1$. Finally, two constants pair to 1 at every point. This proves Weil reciprocity on $\mathbb{P}^1_k$.

From an arbitrary curve to the line

To descend from a general curve to $\mathbb{P}^1$ we use that the tame symbol is compatible with pushing forward along a finite map. Let $\pi : C' \to C$ be the finite morphism of smooth projective curves corresponding to a finite extension L/K of function fields, and let $q \in C'$ lie over $p \in C$.

For $a \in K^\times$ and $b \in L^\times$,

$$\prod_{q|p} N_{k(q)/k(p)}\,(a,b)_q = (a,\ N_{L/K}b)_p.$$

Proof. This is a local statement about a finite extension of discretely valued fields, and by bimultiplicativity it suffices to treat a a unit and a a uniformizer.

Suppose first that a is a unit at p, hence a unit at every $q \mid p$. Writing $n_q = v_q(b)$, the unit formula gives $(a,b)_q = \overline{a}^{\,n_q}$ in $k(q)^\times$, where $\overline{a}$ is the residue of a. Taking norms and multiplying over $q \mid p$,

$$\prod_{q|p} N_{k(q)/k(p)}(\overline{a}^{\,n_q}) = \overline{a}^{\ \sum_{q|p} n_q [k(q):k(p)]}.$$

The exponent is exactly the valuation of the field norm,

$$v_p(N_{L/K}b) = \sum_{q|p} v_q(b)\,[k(q):k(p)],$$

which is the standard formula relating the valuation of a norm to the local degrees. So the product equals $\overline{a}^{\,v_p(N_{L/K}b)}$, and since a is a unit this is precisely $(a, N_{L/K}b)_p$.

Suppose next that $a = \pi_p$ is a uniformizer at p. At each $q \mid p$ write $\pi_p = u_q \pi_q^{e_q}$ with π_q a uniformizer, e_q the ramification index, and u_q a unit. For b a unit at q, the unit formula gives $(\pi_p, b)_q = b(q)^{-e_q}$; taking norms over $q \mid p$ and multiplying assembles the residue of $N_{L/K}(b)^{-1}$ at p, which is $(\pi_p, N_{L/K}b)_p$. For $b = \pi_q$, the standard local norm computation gives that $N_{L_q/K_p}(\pi_q)$ has valuation $[k(q):k(p)]$ with residue unit accounted for by the norm of the residue of u_q and the sign, matching $(\pi_p, N_{L/K}b)_p$ term by term. Bimultiplicativity in b combines the unit and uniformizer cases, and since every $a \in K^\times$ is a unit times a power of a uniformizer, the lemma follows. □

Now let C be any smooth projective curve over an algebraically closed k, and take $f, g \in k(C)^\times$. If f is constant equal to c, then Weil reciprocity follows directly from the degree formula:

$$\prod_p (c,g)_p = \prod_p c^{\,v_p(g)} = c^{\ \sum_p v_p(g)} = c^0 = 1.$$

If f is nonconstant, it defines a finite morphism $f : C \to \mathbb{P}^1$, equivalently an inclusion $k(t) \hookrightarrow k(C)$ with $t = f$. Apply the norm compatibility lemma with $a = t \in k(t)^\times$ and $b = g \in k(C)^\times$, and multiply over all points $x \in \mathbb{P}^1$:

$$\prod_{p\in C} (f,g)_p = \prod_{x\in\mathbb{P}^1} \prod_{p\mapsto x} (t,g)_p = \prod_{x\in\mathbb{P}^1} (t,\ N_{k(C)/k(t)}g)_x.$$

The right-hand side is a product of tame symbols on $\mathbb{P}^1$, which is 1 by the case already proved. Hence $\prod_p (f,g)_p = 1$.

For a general base field k, the same argument runs with residue degrees carried along: the symbol at p lives in $k(p)^\times$, one applies $N_{k(p)/k}$ before multiplying, and the norm compatibility lemma is stated exactly so that these residue norms match up. This gives the general form $\prod_p N_{k(p)/k}(f,g)_p = 1$.

Notice the structure of the proof. The line is handled by an explicit finite computation, and every other curve is reduced to the line by pushing forward along the map f itself. This is the same two-step pattern, direct check on $\mathbb{P}^1$ plus norm compatibility, that proves the residue theorem and the degree formula, and it is the geometric reason all three laws hold.

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4. The K-theoretic interpretation

The tame symbol is a boundary map, and naming it as such places Weil reciprocity in a pattern that continues to all higher reciprocity laws.

We discuss just Milnor K-theory so as to not break our stride. For a field F, the group $K_1^M(F)$ is just $F^\times$, and $K_2^M(F)$ is generated by symbols $\{a,b\}$ with $a,b \in F^\times$, subject to bilinearity and the Steinberg relation $\{a,1-a\}=0$ for $a \neq 0,1$. A symbol $\{a,b\}$ should be thought of as a formal, antisymmetric, bilinear pairing of units, and the Steinberg relation is the one nontrivial law it obeys.

Now let F be a field with a discrete valuation v and residue field κ. There are boundary maps recording how a K-class degenerates at the closed point. In degree one, the boundary is the valuation itself,

$$\partial : K_1^M(F) = F^\times \longrightarrow \mathbb{Z} = K_0^M(\kappa), \qquad \partial(f) = v(f),$$

which sends a function to its order of vanishing. In degree two, the boundary is the tame symbol,

$$\partial : K_2^M(F) \longrightarrow \kappa^\times = K_1^M(\kappa), \qquad \partial\{f,g\} = (f,g)_v.$$

The Steinberg relation proved in Section 2 is exactly the statement that this map is well defined: since $(f,1-f)_v = 1$, the tame symbol kills the defining relations of K_2^M and descends to the quotient.

On a curve C, every closed point p supplies such a boundary map $\partial_p : K_2^M(k(C)) \to k(p)^\times$. Weil reciprocity is then the statement that the total boundary, taken over all points and pushed down to k by the residue norms, is trivial:

$$\prod_{p \in C} N_{k(p)/k}\, \partial_p\{f,g\} = 1.$$

This is the K_2 analogue of the degree formula, which is the same statement one rung down:

$$\sum_{p\in C}[k(p):k]\,\partial_p(f)=\sum_{p\in C}v_p(f)\,[k(p):k]=0.$$

The table below lucidly illustrates the parallel.

Group	**Local boundary at p**	**Global relation**
$K_1^M(k(C))=k(C)^\times$	$v_p(f)\in\mathbb{Z}$	$\sum_p[k(p):k]\,v_p(f)=0$
$K_2^M(k(C))$	$\partial_p\{f,g\}\in k(p)^\times$	$\prod_p N_{k(p)/k}\,\partial_p\{f,g\}=1$

Both rows say that the sum, or product, of local boundaries of a global K-class vanishes. This is why Weil reciprocity is properly seen as a first higher reciprocity law rather than an isolated identity: it is the degree-two case of a phenomenon that has a case in every degree. The pattern continues to higher-dimensional schemes and higher Milnor K-groups, where the boundary maps assemble into the Gersten complex and the analogous total-boundary vanishing is the foundation of higher-dimensional class field theory in the work of Parshin and of Kato and Saito. We will not need said higher theory, but it is the same idea: a global class has no net local boundary.

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5. Function fields over finite fields

When the base is a finite field the geometric picture becomes literal arithmetic. Let $X/\mathbb{F}_q$ be a smooth projective geometrically integral curve with function field $K=\mathbb{F}_q(X)$. The closed points of X are precisely the places of the global field K, and the residue field at a closed point x is a finite extension $k(x)/\mathbb{F}_q$ of degree $\deg x=[k(x):\mathbb{F}_q]$.

The degree formula becomes a weighted sum over places,

$$\sum_x v_x(f)\ \deg x=0,$$

and if we package the local data into an absolute value $|f|_x=q^{-\deg(x)\,v_x(f)}$, this is exactly the product formula

$$\prod_x|f|_x=1$$

for the global field $\mathbb{F}_q(X)$. Weil reciprocity becomes the arithmetic statement

$$\prod_x N_{k(x)/\mathbb{F}_q}\,(f,g)_x=1,$$

now a genuine identity in the multiplicative group of the finite field $\mathbb{F}_q$.

Class field theory also has a function-field incarnation here. Writing $\mathbb{A}_K^\times$ for the idele group and $C_K = \mathbb{A}_K^\times / K^\times$ for the idele class group, global class field theory gives a reciprocity map $C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ under which, at an unramified closed point x, a local uniformizer maps to a Frobenius element, subject to the usual convention fixing arithmetic versus geometric Frobenius. The single curve X thereby carries valuations, a product formula, tame symbols, Frobenius elements, and abelian Galois theory, all at once. The finite-field case is where the analogy stops being an analogy.

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6. Number fields as arithmetic curves

Let K be a number field with ring of integers $\mathcal{O}_K$. Its finite places are the nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}_K$, each giving a valuation $v_\mathfrak{p} : K^\times \to \mathbb{Z}$, and its archimedean places come from the real embeddings and the conjugate pairs of complex embeddings. Together these are the places of K, the arithmetic analogue of the closed points of a curve, with the archimedean places supplying the points at infinity.

Normalize the absolute values so that the product formula will hold. At a finite place, set $|a|_\mathfrak{p} = N(\mathfrak{p})^{-v_\mathfrak{p}(a)}$, where $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$. At a real place given by $\sigma : K \hookrightarrow \mathbb{R}$, set $|a|_\sigma = |\sigma(a)|$. At a complex place represented by $\tau : K \hookrightarrow \mathbb{C}$, set $|a|_\tau = |\tau(a)|^2$, the square appearing because a complex place carries two embeddings.

Product formula. For every $a \in K^\times$,

$$\prod_v |a|_v = 1.$$

Proof. Factor the principal fractional ideal into primes, $(a) = \prod_\mathfrak{p} \mathfrak{p}^{\,v_\mathfrak{p}(a)}$, and take absolute norms. Since the norm is multiplicative and $N(\mathfrak{p}^n) = N(\mathfrak{p})^n$,

$$|N_{K/\mathbb{Q}}(a)| = N\big((a)\big) = \prod_\mathfrak{p} N(\mathfrak{p})^{\,v_\mathfrak{p}(a)}.$$

The product of the finite absolute values is therefore the reciprocal of this:

$$\prod_\mathfrak{p} |a|_\mathfrak{p} = \prod_\mathfrak{p} N(\mathfrak{p})^{-v_\mathfrak{p}(a)} = |N_{K/\mathbb{Q}}(a)|^{-1}.$$

On the archimedean side, the field norm is the product of all embeddings, which groups into real embeddings and conjugate complex pairs:

$$|N_{K/\mathbb{Q}}(a)| = \prod_{\sigma : K \hookrightarrow \mathbb{R}} |\sigma(a)| \prod_{\tau : K \hookrightarrow \mathbb{C}/\,\mathrm{conj}} |\tau(a)|^2,$$

which is exactly the product of the archimedean absolute values. Multiplying the finite contribution $|N_{K/\mathbb{Q}}(a)|^{-1}$ by the archimedean contribution $|N_{K/\mathbb{Q}}(a)|$ gives 1. ☐

This is the number-field form of $\deg \operatorname{div}(f) = 0$. The finite places contribute the ideal-theoretic norm, the archimedean places contribute the analytic norm, and their product is one precisely because $\operatorname{Spec} \mathcal{O}_K$ together with its points at infinity is a complete arithmetic curve. The refinements of this law, the arithmetic analogues of Weil reciprocity, are the reciprocity laws of number theory.

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7. Hilbert reciprocity

Hilbert reciprocity is the quadratic analogue of Weil reciprocity for number fields. Where Weil reciprocity pairs two functions through the tame symbol, Hilbert reciprocity pairs two numbers through a symbol valued in $\{\pm 1\}$, and the global law again asserts that the product over all places is trivial.

The local pairing is the Hilbert symbol. Let F be a local field of characteristic different from 2. For $a, b \in F^\times$, the Hilbert symbol $(a, b)_F \in \{\pm 1\}$ can be described in either of two equivalent ways:

- $(a, b)_F = 1$ if and only if a is a norm from the extension $F(\sqrt{b})$, meaning $a = N_{F(\sqrt{b})/F}(z)$ for some z;
- $(a, b)_F = 1$ if and only if the quaternion algebra with generators i, j satisfying $i^2 = a$, $j^2 = b$, and $ij = -ji$ is split over F, that is, isomorphic to the 2×2 matrix algebra.

The two descriptions agree because a quaternion algebra is split exactly when the norm form of $F(\sqrt{b})$ represents a, which is the theory of quaternion algebras over a field. The symbol is symmetric and bimultiplicative, and it records whether the conic $ax^2 + by^2 = z^2$ has a nontrivial F-point.

Now let K be a number field. For each place v, apply this construction over the completion K_v and write $(a, b)_v$ for the resulting Hilbert symbol.

Hilbert reciprocity. For $a, b \in K^\times$,

$$\prod_v (a, b)_v = 1,$$

the product over all places of K, in which all but finitely many factors are 1.

A complete proof of this is one of the classical forms of global class field theory. The most transparent route is through the Brauer group, which we now describe, taking the Brauer–Hasse–Noether theorem as the input from class field theory.

The quaternion algebra (a,b) is a central simple algebra over K, so it defines a class in the Brauer group $\mathrm{Br}(K)$, and since it is split by a quadratic extension its class lies in the 2-torsion $\mathrm{Br}(K)[2]$. For each place v, base change to K_v gives a local class in $\mathrm{Br}(K_v)$, and local class field theory provides an invariant isomorphism

$$\mathrm{inv}_v : \mathrm{Br}(K_v) \hookrightarrow \mathbb{Q}/\mathbb{Z},$$

under which a quaternion algebra, being 2-torsion, has invariant 0 when it splits and $\frac{1}{2}$ when it is a division algebra. The global and local Brauer groups are tied together by the fundamental exact sequence of global class field theory.

Brauer–Hasse–Noether. For a number field K, the sequence

$$0 \to \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact. In particular, the local invariants of a global Brauer class sum to zero.

Applying the sum-of-invariants map to the global class $(a,b) \in \mathrm{Br}(K)$ gives

$$\sum_v \mathrm{inv}_v\big((a,b) \otimes_K K_v\big) = 0 \quad \text{in } \mathbb{Q}/\mathbb{Z}.$$

Each local invariant is 0 or $\frac{1}{2}$, so the number of places at which the local quaternion algebra fails to split is even. Translating $\frac{1}{2}$ into the sign -1 and 0 into $+1$, the product of the local Hilbert symbols is $\prod_v (a,b)_v = 1$. This is the direct bridge from Hilbert reciprocity to the Brauer–Manin obstruction: a global Brauer class has local invariants whose total is zero, and any failure of a global point to exist is measured against exactly this constraint.

Quadratic reciprocity as a special case

The reason Hilbert reciprocity deserves the name is that it contains the classical quadratic reciprocity law of Gauss. Take $K = \mathbb{Q}$, whose places are the primes p together with the archimedean place ∞. The local Hilbert symbols over $\mathbb{Q}$ have explicit formulas, which we state and then use; each is a short local computation of when a conic has points.

For an odd prime p and a p-adic unit u, the symbol $(u,p)_p = \left(\frac{u}{p}\right)$ is the Legendre symbol of the residue of u, since p is a norm from $\mathbb{Q}_p(\sqrt{u})$ exactly when u is a square modulo p. For two odd 2-adic units u, w, the symbol at 2 is $(u,w)_2 = (-1)^{\frac{u-1}{2}\frac{w-1}{2}}$. At the real place, $(a,b)_\infty = -1$ exactly when a and b are both negative, since the conic $ax^2 + by^2 = z^2$ has a real point unless $a, b < 0$.

Let p and q be distinct odd primes, and apply Hilbert reciprocity to $a = p$ and $b = q$. We evaluate every local symbol.

At a finite place $\ell \notin \{p, q\}$, both p and q are units, and a product of units gives a split conic, so $(p, q)_\ell = 1$.

At the place p, the number q is a unit, and the symbol is $(p, q)_p = (q, p)_p = \left(\frac{q}{p}\right)$ by the odd-prime formula.

At the place q, symmetrically, $(p, q)_q = \left(\frac{p}{q}\right)$.

At the place 2, both p and q are odd units, so $(p, q)_2 = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$.

At the place ∞, both p and q are positive, so $(p, q)_\infty = 1$.

Hilbert reciprocity says the product of all these is 1:

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)(-1)^{\frac{p-1}{2}\frac{q-1}{2}} = 1.$$

Rearranging, and using that Legendre symbols are ± 1,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

which is exactly Gauss's law of quadratic reciprocity. The two supplementary laws arise the same way: applying Hilbert reciprocity to $(-1, p)$ recovers $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, and to $(2, p)$ recovers $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$. The mysterious symmetry between p and q in quadratic reciprocity is, from this vantage, the statement that a single global object, the quaternion algebra (p, q) over $\mathbb{Q}$, has local invariants summing to zero.

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8. Artin reciprocity

Artin reciprocity is the complete abelian reciprocity law, of which the divisor formula, Weil reciprocity, and Hilbert reciprocity are all shadows. It describes every finite abelian extension of a global field at once.

Let K be a global field, either a number field or a function field over a finite field. Its idele group $\mathbb{A}_K^\times$ is the restricted product of the local groups $K_v^\times$, and its idele class group is $C_K = \mathbb{A}_K^\times / K^\times$. Global class field theory constructs the Artin reciprocity map

$$\mathrm{Art}_K : C_K \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

a continuous homomorphism to the abelianized absolute Galois group, and for each finite abelian extension L/K it induces an isomorphism

$$C_K / N_{L/K} C_L \;\cong\; \mathrm{Gal}(L/K),$$

with the usual convention fixing the choice between arithmetic and geometric Frobenius. The norm subgroup $N_{L/K} C_L$ is the kernel that cuts C_K down to the finite Galois group.

The local information the map carries is Frobenius. At a finite place v unramified in L, the idele that equals a uniformizer ϖ_v at v and 1 at every other place maps to the Frobenius element at v:

$$\mathrm{Art}_K(\dots, 1, \varpi_v, 1, \dots) = \mathrm{Frob}_v \in \mathrm{Gal}(L/K).$$

So the Artin map assembles all local Frobenius elements into a single global homomorphism, and the reciprocity content is what happens on principal ideles.

The quotient by $K^\times$ is the global constraint. Every principal idele, the image of an element $a \in K^\times$ under the diagonal embedding, lies in the kernel of Art_K and therefore acts trivially in every abelian Galois group. This is the exact analogue, one categorical level up, of the earlier laws: there the global object was a rational function or a number, and the constraint was that its total local boundary vanished; here the global object is a principal idele, and the constraint is that it acts trivially on abelian extensions.

Reciprocity law	**Local object**	**Global constraint**
divisor formula	valuations $v_p(f)$	total degree 0
Weil reciprocity	tame symbols $(f, g)_p$	product 1
Hilbert reciprocity	Hilbert symbols $(a, b)_v$	product 1
Artin reciprocity	local uniformizers and units	principal ideles act trivially

Each row attaches a local invariant at every place and asserts a single global relation. Hilbert reciprocity is the piece of Artin reciprocity visible through quadratic extensions, since the Hilbert symbol measures splitting in the abelian extension $K(\sqrt{b})/K$ and its product formula is the triviality of principal ideles for that extension. Weil reciprocity is the analogous statement for the function field of a curve with the multiplicative group as coefficients. Artin reciprocity is the endpoint that contains them, the full class-field-theoretic form of the local-to-global principle.

Denouement

A picture (or a chart!) tells a thousand words.

Complex or algebraic curve	**Finite-field curve**	**Number field**
compact curve C/k	$X/\mathbb{F}_q$	Spec $\mathcal{O}_K$ plus infinity
$k(C)$	$\mathbb{F}_q(X)$	K
closed point p	closed point x	place v
valuation v_p	valuation v_x	v or $\lvert \cdot \rvert_v$
$\deg \operatorname{div}(f) = 0$	$\prod_x \lvert f \rvert_x = 1$	$\prod_v \lvert a \rvert_v = 1$
residue theorem	tame reciprocity	Hilbert reciprocity
Weil reciprocity	Weil reciprocity	Artin reciprocity
Jacobians and divisors	ideles and Frobenius	ideles and Galois groups