p-adic representations and period rings

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> *Three rings for the Elven-kings under the sky,*
> $B_{\text{cris}},\ B_{\text{st}},\ B_{\text{dR}}$
> *Seven for the Dwarf-lords in their halls of stone,*
> $\widetilde{A},\ E_{\mathbb{Q}_p},\ A_{\mathbb{Q}_p},\ B_{\mathbb{Q}_p},\ E,\ A,\ B,$
> *Nine for the mortal Men doomed to die,*
> $\mathbb{Q}_p,\ \mathbb{Z}_p,\ \mathbb{F}_p,\ \overline{\mathbb{Q}}_p,\ \mathbb{C}_p,\ \mathcal{O}_{\mathbb{C}_p},\ \mathbb{Q}_p^{\text{unr}},\ B_{\text{HT}}, \widehat{\mathbb{Q}_p^{\text{unr}}}$
> *One ring to rule them all,*
> $\mathbb{A}_{\text{inf}}.$
>
> — P. Colmez

The goal of this post is to introduce the first layer of p-adic Hodge theory: p-adic Galois representations and Fontaine's period rings. The guiding question is: what extra structure is carried by p-adic Galois representations that arise geometrically?

Let $K/\mathbb{Q}_p$ be a finite extension, and let $G_K = \text{Gal}(\overline{K}/K)$ be its absolute Galois group. A p-adic representation of G_K is a finite-dimensional $\mathbb{Q}_p$-vector space V equipped with a continuous action of G_K. Smooth proper varieties over K produce such representations through their étale cohomology, $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

The central paradigm of p-adic Hodge theory is that these geometric representations carry Hodge-theoretic, de Rham, crystalline, and semistable structures. Like a prism splitting white light into its constituent colors, Fontaine's period rings are the coefficient rings that reveal those structures. The path runs from p-adic Galois representations, to invariants defined using period rings, to linear algebra carrying a filtration, a Frobenius, and a monodromy operator. This post focuses on the first two steps: the representations, and the period rings themselves.

A p-adic Galois representation is a topological object: a continuous action of a profinite group on a p-adic vector space. Algebraic de Rham cohomology is a linear-algebraic object: a filtered vector space over K. $A\ priori$, these live in different worlds, and there is no map between them until one enlarges the coefficients. The period rings are precisely the enlargement that lets the two worlds communicate. Each period ring is a $\mathbb{Q}_p$-algebra carrying a G_K-action together with some extra structure (a grading, a filtration, a Frobenius, a monodromy operator), and each is designed so that its invariants extract exactly one of those extra structures from a representation.

Later posts will study the linear algebra that emerges, in particular filtered φ-modules, weak admissibility, isocrystals, slopes, the construction of the Fargues–Fontaine curve, the classification of vector bundles on it, and the appearance of geometric local Langlands.

Local fields

We start by reviewing basic ideas on local fields.

Let $K/\mathbb{Q}_p$ be a finite extension. Its ring of integers is

$$\mathcal{O}_K = \{x \in K : |x|_p \leq 1\},$$

its maximal ideal is $\mathfrak{m}_K = \{x \in K : |x|_p < 1\}$, and its residue field is $k = \mathcal{O}_K/\mathfrak{m}_K$. Since $K/\mathbb{Q}_p$ is finite, k is a finite field, and we write $k \cong \mathbb{F}_q$. We choose a uniformizer $\pi \in \mathcal{O}_K$, so that $\mathfrak{m}_K = (\pi)$, and we normalize the valuation $v_K : K^\times \to \mathbb{Z}$ by $v_K(\pi) = 1$. The ramification index and residue degree of $K/\mathbb{Q}_p$ are then defined by $e = v_K(p)$ and $q = p^f$. These two invariants recover the degree through the identity $[K : \mathbb{Q}_p] = ef$.

Proof. The ring $\mathcal{O}_K$ is a discrete valuation ring: K is a finite extension of the complete discretely valued field $\mathbb{Q}_p$, and the valuation extends uniquely to K with value group $\frac{1}{e}\mathbb{Z} \cong \mathbb{Z}$. Since $\mathcal{O}_K$ is finite as a module over the complete DVR $\mathbb{Z}_p$ and is torsion-free, it is free, say of rank $d = [K : \mathbb{Q}_p]$. We produce an explicit $\mathbb{Z}_p$-basis and count.

Choose elements $\omega_1, \ldots, \omega_f \in \mathcal{O}_K$ whose reductions $\overline{\omega}_1, \ldots, \overline{\omega}_f$ form an $\mathbb{F}_p$-basis of k; this is possible because $[k : \mathbb{F}_p] = f$. We claim the ef elements

$$\pi^a \omega_b, \qquad 0 \leq a < e,\ 1 \leq b \leq f,$$

form a $\mathbb{Z}_p$-basis of $\mathcal{O}_K$. For spanning, take $x \in \mathcal{O}_K$. Reducing modulo $\mathfrak{m}_K$ and lifting, we may write $x = \sum_b c_b^{(0)} \omega_b + \pi x_1$ with $c_b^{(0)} \in \mathbb{Z}_p$ representing residues and $x_1 \in \mathcal{O}_K$; iterating on x_1 and collecting powers of π that exceed $e - 1$ into higher powers of p (using $v_K(p) = e$, so $\pi^e = p \cdot (\text{unit})$), the p-adic completeness of $\mathcal{O}_K$ lets the process converge and expresses x as a

$\mathbb{Z}_p$-combination of the listed elements. For independence, a nontrivial relation of minimal p-adic valuation among the coefficients would, after dividing by the appropriate power of p, reduce to a nontrivial $\mathbb{F}_p$-relation among the $\pi^a \overline{\omega}_b$ in the graded pieces $\mathfrak{m}_K^a / \mathfrak{m}_K^{a+1}$, contradicting the choice of the ω_b and the fact that $1, \pi, \ldots, \pi^{e-1}$ have distinct valuations modulo e. Hence the ef elements form a basis, and $d = ef$. □

Let $W(k)$ be the ring of Witt vectors of k, and set $K_0 = W(k)[1/p]$. Then $K_0/\mathbb{Q}_p$ is the unique unramified extension of degree f. After fixing an embedding $K_0 \hookrightarrow K$, it is the maximal unramified subextension of K, and K/K_0 is totally ramified. The field K_0 carries a Frobenius automorphism $\sigma : K_0 \to K_0$ lifting the p-power map $x \mapsto x^p$ on k.

Proof. For a perfect field k of characteristic p, the Witt vector ring $W(k)$ is the unique p-adically complete discrete valuation ring with uniformizer p and residue field k; this is the defining universal property of p-typical Witt vectors. Since $k/\mathbb{F}_p$ is finite of degree f, the extension $W(k)[1/p]$ over $\mathbb{Q}_p = W(\mathbb{F}_p)[1/p]$ is unramified (its uniformizer is still p) of residue degree f, hence of degree f, and it is the unique such extension because unramified extensions correspond bijectively to residue-field extensions.

The Frobenius σ is obtained by functoriality: the absolute Frobenius $\mathrm{Fr}_k : x \mapsto x^p$ on k is a ring homomorphism, and $W(-)$ is a functor, so $W(\mathrm{Fr}_k)$ is a ring endomorphism of $W(k)$ reducing to Fr_k modulo p. Since k is finite, Fr_k is an automorphism, so $\sigma = W(\mathrm{Fr}_k)$ is an automorphism, and it inverts to $W(\mathrm{Fr}_k^{-1})$. Inverting p extends σ to K_0. □

Let $\overline{K}$ be an algebraic closure of K, and define $C = \widehat{\overline{K}}$, the completion of $\overline{K}$ for the unique extension of the p-adic absolute value. The field C is complete and nonarchimedean by construction, and it is algebraically closed. The absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ is a profinite group, and its action on $\overline{K}$ is isometric, hence extends continuously to C.

Proof. The group G_K is the inverse limit of the finite quotients $\mathrm{Gal}(L/K)$ as L ranges over finite Galois extensions of K inside $\overline{K}$, so it is profinite. Each $g \in G_K$ preserves the absolute value on $\overline{K}$ (the extension of $|\cdot|_p$ to $\overline{K}$ is unique, so it is Galois-invariant), hence g is uniformly continuous and extends uniquely to the completion C.

For algebraic closedness, we use Krasner's lemma. Let $f(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_0 \in C[X]$ be monic; we show it has a root in C. Since $\overline{K}$ is dense in C, choose $\widetilde{c}_i \in \overline{K}$ with $|\widetilde{c}_i - c_i|$ small, and set $\widetilde{f}(X) = X^n + \widetilde{c}_{n-1}X^{n-1} + \cdots + \widetilde{c}_0$. Then $\widetilde{f}$ splits over $\overline{K}$, say with roots $\beta_1, \ldots, \beta_n$. The roots of f and $\widetilde{f}$ are close: if α is any root of f then $|\widetilde{f}(\alpha)| = |\widetilde{f}(\alpha) - f(\alpha)|$ is small, and since $\widetilde{f}(\alpha) = \prod_j (\alpha - \beta_j)$, some β_j

satisfies $|\alpha - \beta_j|$ small. Making the approximation fine enough that $|\alpha - \beta_j|$ is smaller than $|\alpha - \alpha'|$ for every other root $\alpha' \neq \alpha$ of f, Krasner's lemma gives $K(\alpha) \subseteq K(\beta_j) \subseteq \overline{K}$; since α was already algebraic over C and now lies within $\overline{K}$-distance tending to 0, the sequence of such β_j converges in C to α, so $\alpha \in C$. Hence every monic polynomial over C has all roots in C. □

Inertia and Frobenius

Let $K^{\mathrm{ur}} \subset \overline{K}$ be the maximal unramified extension of K, and let $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{ur}})$ be the inertia subgroup. Reduction to the residue field gives an exact sequence

$$1 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

Since $k \cong \mathbb{F}_q$, the quotient is $\mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$, topologically generated by the arithmetic Frobenius $\mathrm{Frob}_q^{\mathrm{arith}} : x \mapsto x^q$; its inverse is the geometric Frobenius.

Proof. Let $\mathcal{O}_{\overline{K}}$ be the integral closure of $\mathcal{O}_K$ in $\overline{K}$. It is a valuation ring (nondiscrete) with maximal ideal $\mathfrak{m}_{\overline{K}}$, and its residue field is $\overline{k}$: any element of $\overline{k}$ is a root of a polynomial over k, which lifts to a monic polynomial over $\mathcal{O}_K$ whose roots lie in $\mathcal{O}_{\overline{K}}$, so the residue field is algebraically closed over k, hence equals $\overline{k}$. Each $g \in G_K$ preserves $\mathcal{O}_{\overline{K}}$ and $\mathfrak{m}_{\overline{K}}$, so it induces an automorphism $\overline{g}$ of $\overline{k}$ fixing k. The assignment $g \mapsto \overline{g}$ is a continuous homomorphism $G_K \to \mathrm{Gal}(\overline{k}/k)$.

The kernel consists of those g acting trivially on $\overline{k}$. An automorphism acts trivially on the residue field if and only if it fixes every unramified subextension, that is, if and only if it lies in $\mathrm{Gal}(\overline{K}/K^{\mathrm{ur}}) = I_K$. Thus the kernel is I_K.

For surjectivity, recall that finite unramified extensions of K correspond bijectively to finite (necessarily separable, as k is perfect) extensions of k, with $\mathrm{Gal}(K'/K) \cong \mathrm{Gal}(k'/k)$ for the corresponding fields. Given any element of $\mathrm{Gal}(\overline{k}/k)$, its restrictions to finite k'/k lift to compatible elements of $\mathrm{Gal}(K'/K)$, and taking the inverse limit produces a preimage in G_K. Finally, $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$, with the topological generator $x \mapsto x^q$, because the finite quotients $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$ are generated by $x \mapsto x^q$ compatibly in n. □

Thus, G_K carries two fundamental types of information: inertia, measuring ramification, and Frobenius, measuring residue-field arithmetic. This dichotomy reappears throughout local arithmetic. In ℓ-adic cohomology for $\ell \neq p$, unramified representations are largely controlled by Frobenius, and inertia acts through a finite quotient on a semisimplified representation. In p-adic Hodge theory, the situation is more delicate, as inertia can act through infinite quotients in ways that encode Hodge filtrations and monodromy. Rendering precise that encoding is the purpose of the period rings.

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p-adic representations

A p-adic representation of G_K is a finite-dimensional $\mathbb{Q}_p$-vector space V equipped with a continuous action $G_K \times V \to V$. Equivalently, it is a continuous homomorphism $\rho : G_K \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ carries its p-adic topology.

After choosing a basis of V, a p-adic representation of dimension n is a continuous homomorphism $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_p)$. We denote the category of p-adic representations of G_K by $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. It is a tensor category: indeed, if $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, then so are $V \oplus W$, $V \otimes_{\mathbb{Q}_p} W$, $V^\vee$, and $\mathrm{Hom}(V, W)$.

Proof. For the direct sum, set $g(v, w) = (gv, gw)$. For the tensor product, set $g(v \otimes w) = gv \otimes gw$ on simple tensors and extend $\mathbb{Q}_p$-linearly; this is well defined because g acts $\mathbb{Q}_p$-linearly on each factor. For the dual, set $(g\lambda)(v) = \lambda(g^{-1}v)$, which is a left action because $(gh)^{-1} = h^{-1}g^{-1}$. For $\mathrm{Hom}(V, W) \cong V^\vee \otimes W$, this specializes to $(gf)(v) = g\big(f(g^{-1}v)\big)$.

Continuity holds for each construction. Each of $\oplus, \otimes, (-)^\vee, \mathrm{Hom}$ is a continuous functor on finite-dimensional p-adic vector spaces (the relevant maps on matrix entries are polynomial in the entries of the constituent representations, and inversion is continuous on GL_n), so composing ρ_V and ρ_W with these operations yields continuous homomorphisms. $\square$

The trivial representation is $\mathbb{Q}_p$, with G_K acting trivially.

Continuity is a substantive condition. Since G_K is compact and ρ is continuous, the image $\rho(G_K)$ is a compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Compactness is what forces the existence of stable lattices, which in turn bind p-adic representations to finite arithmetic objects.

Lemma. Let V be a p-adic representation of G_K. Then there exists a G_K-stable $\mathbb{Z}_p$-lattice $T \subset V$, meaning a finitely generated $\mathbb{Z}_p$-submodule with $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ and $gT = T$ for every $g \in G_K$.

Proof. Fix an isomorphism $V \cong \mathbb{Q}_p^n$ and let $T_0 = \mathbb{Z}_p^n$ be the standard lattice, with stabilizer $\mathrm{GL}_n(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)$. The subgroup $\mathrm{GL}_n(\mathbb{Z}_p)$ is open and compact in $\mathrm{GL}_n(\mathbb{Q}_p)$, so its preimage

$$H_0 = \rho^{-1}\big(\mathrm{GL}_n(\mathbb{Z}_p)\big) \subseteq G_K$$

is open. Since G_K is compact, H_0 has finite index; choose coset representatives $g_1, \dots, g_m$ for G_K/H_0. Define

$$T = \sum_{i=1}^{m} g_i T_0,$$

a finite sum of $\mathbb{Z}_p$-lattices, hence itself a $\mathbb{Z}_p$-lattice in V (finitely generated, and spanning V over $\mathbb{Q}_p$ since it contains T_0).

We check G_K-stability. Let $g \in G_K$. For each i, the product gg_i lies in some coset, say $gg_i \in g_{\pi(i)}H_0$, where π is a permutation of $\{1, \dots, m\}$ (left multiplication permutes cosets). Writing $gg_i = g_{\pi(i)}h_i$ with $h_i \in H_0$, and noting $\rho(h_i)T_0 = T_0$ by definition of H_0, we get $gg_iT_0 = g_{\pi(i)}h_iT_0 = g_{\pi(i)}T_0$. Therefore

$$gT = \sum_i gg_iT_0 = \sum_i g_{\pi(i)}T_0 = \sum_j g_jT_0 = T.$$

Thus T is G_K-stable. □

Such a lattice allows one to approximate V by the finite Galois modules T/p^nT. This is one reason p-adic representations connect both to continuous p-adic linear algebra and to finite arithmetic objects, and it is the mechanism behind reduction modulo p of Galois representations.

The cyclotomic character

For every $n \geq 1$, let $\mu_{p^n} \subset \overline{K}^\times$ be the group of p^n-th roots of unity. The group G_K acts on μ_{p^n}, and after choosing a primitive root ζ_{p^n}, every $g \in G_K$ satisfies $g(\zeta_{p^n}) = \zeta_{p^n}^{a_n(g)}$ for a unique $a_n(g) \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. The numbers $a_n(g)$ are compatible as n varies, and passing to the inverse limit produces the p-adic cyclotomic character $\chi_{\text{cyc}} : G_K \to \mathbb{Z}_p^\times$, characterized by $g(\zeta) = \zeta^{\chi_{\text{cyc}}(g)}$ for all p-power roots of unity ζ.

Proof. The group μ_{p^n} is cyclic of order p^n, so an automorphism is determined by its action on a generator, and it must send a generator to a generator. Hence $\text{Aut}(\mu_{p^n}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$, and the Galois action gives a homomorphism $\chi_n : G_K \to (\mathbb{Z}/p^n\mathbb{Z})^\times$. This homomorphism is continuous: its kernel is $\text{Gal}(\overline{K}/K(\mu_{p^n}))$, which is open because $K(\mu_{p^n})/K$ is finite.

The transition maps $\mu_{p^{n+1}} \to \mu_{p^n}$, $\zeta \mapsto \zeta^p$, are G_K-equivariant, so the χ_n are compatible with the reduction maps $(\mathbb{Z}/p^{n+1})^\times \to (\mathbb{Z}/p^n)^\times$. Taking the inverse limit gives a continuous homomorphism

$$\chi_{\text{cyc}} = \varprojlim_n \chi_n : G_K \to \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

The defining identity $g(\zeta) = \zeta^{\chi_{\text{cyc}}(g)}$ holds level by level, hence for all p-power roots of unity. □

The representation $\mathbb{Q}_p(1)$ is the one-dimensional $\mathbb{Q}_p$-vector space $\mathbb{Q}_p \cdot e$ on which G_K acts by the cyclotomic character, $g \cdot e = \chi_{\text{cyc}}(g)e$. For $n \in \mathbb{Z}$, define $\mathbb{Q}_p(n) = \mathbb{Q}_p(1)^{\otimes n}$, with the convention $\mathbb{Q}_p(-n) = \mathbb{Q}_p(n)^\vee$ for $n > 0$.

These are the Tate twists. If V is a p-adic representation, define $V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$. Tate twists shift weights, and they appear naturally in cohomology because the cycle class of a subvariety of codimension j lives in cohomology with a $\mathbb{Q}_p(-j)$ twist. The cleanest instance is projective space:

$$H^{2i}_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-i) \quad (0 \le i \le n), \qquad H^{2i+1}_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p) = 0.$$

Proof. By the projective bundle formula in étale cohomology (applied to $\mathbb{P}^n = \mathbb{P}(V_0)$ for an $(n+1)$-dimensional vector space), the total cohomology ring is

$$H^\bullet_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p[h]/(h^{n+1}), \qquad h \in H^2_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p(1)),$$

where h is the first Chern class of $\mathcal{O}(1)$, a canonical element of the $\mathbb{Q}_p(1)$-twisted H^2. Thus $H^{2i}_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p(i))$ is one-dimensional, generated by h^i, and G_K acts trivially on this generator (Chern classes are Galois-equivariant and h is defined over K). Untwisting by $\mathbb{Q}_p(-i)$ gives $H^{2i}_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-i)$. The odd cohomology vanishes because the ring $\mathbb{Q}_p[h]/(h^{n+1})$ is concentrated in even degrees.

For the base case $n = 1$, the identification $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-1)$ can also be seen directly: the trace/degree map identifies H^2 with the dual of H^0 twisted by the cycle class of a point, and the Galois action on that class is by χ_{cyc}^{-1}, matching the p-adic Tate module of μ_{p^∞}. □

Thus, $\mathbb{Q}_p(-1)$ is the p-adic cohomology class of a divisor, and it is the p-adic analogue of the class $2\pi i$ in the complex comparison between Betti and de Rham cohomology. Recording where the twists sit is the content of Hodge–Tate theory below.

Representations from geometry

Let X/K be a smooth proper variety. Then p-adic étale cohomology produces finite-dimensional $\mathbb{Q}_p$-vector spaces $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Because G_K acts on $\overline{K}$, it acts on the base change $X_{\overline{K}} = X \times_K \overline{K}$, and therefore on cohomology, making $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ a p-adic representation of G_K.

Proof. For $g \in G_K$, the automorphism $g : \operatorname{Spec} \overline{K} \to \operatorname{Spec} \overline{K}$ induces an automorphism $g_X : X_{\overline{K}} \to X_{\overline{K}}$ of X-schemes covering id_X. Étale cohomology is a contravariant functor, so g_X induces g_X^* on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n)$, and the identity $(gh)_X = g_X h_X$ gives a group action (after passing to a left action via inverses). Passing to the limit over n and inverting p yields a $\mathbb{Q}_p$-linear action on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

Continuity holds because $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) = \left(\varprojlim_n H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, each finite group $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n)$ is a continuous discrete G_K-module (its stabilizers are open, since the cohomology is already defined over some finite extension of K), and an inverse limit of continuous actions is continuous. □

We refer to a p-adic representation V of G_K as geometric, informally, if it occurs as a subquotient of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(n)$ for some smooth proper variety X/K, some degree i, and some Tate twist n.

The precise meaning of “geometric” varies with context (the Fontaine–Mazur conjecture gives a purely Galois-theoretic characterization in the global setting). The relevant point here is that representations arising from geometry satisfy strong regularity properties, which the period rings are built to detect. More precisely, for smooth proper X/K, the representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is always de Rham; if X has good reduction it is crystalline; and if X has semistable reduction it is semistable. Defining these three conditions is the work of the rest of the post.

Why can’t we just extend scalars?

Let X/K be smooth and proper. There are two cohomology theories one would like to compare. The first is étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, a $\mathbb{Q}_p$-vector space carrying a continuous G_K-action. The second is algebraic de Rham cohomology,

$$H^i_{\text{dR}}(X/K) = \mathbb{H}^i(X, \Omega^\bullet_{X/K}),$$

a K-vector space carrying a decreasing Hodge filtration $\operatorname{Fil}^\bullet H^i_{\text{dR}}(X/K)$. The two carry the same numerical information (dimensions agree), and one wants a canonical comparison. Naively extending scalars, however, fails:

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \overset{?}{\cong} H^i_{\text{dR}}(X/K)$$

has no canonical map in either direction.

Explanation. We indicate somewhat imprecisely how the two sides carry incompatible structure. The left-hand side remembers a continuous G_K-action; the right-hand side remembers differential forms and a filtration. Tensoring the left side with K neither creates the Hodge filtration nor introduces differential forms, and it leaves a residual Galois action that the right side does not have. There is no functorial K-linear identification that respects both structures.

The complex-analytic case sheds light on what is missing. For $X/\mathbb{C}$ smooth proper, integration of algebraic forms over topological cycles gives period integrals $\int_\gamma \omega$, and assembling them into a period matrix induces a canonical isomorphism

$$H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X/\mathbb{C}).$$

The field $\mathbb{C}$ is large enough to contain all the periods, and in particular it contains $2\pi i$, the period appearing in the comparison for H^1 of $\mathbb{G}_m$. The base field K in the p-adic setting has no room for such periods: it contains no analogue of $2\pi i$. One therefore enlarges K to a period ring: a $\mathbb{Q}_p$-algebra B with G_K-action, large enough to contain the p-adic periods, so that the comparison takes the form

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B \;\cong\; H^i_{\mathrm{dR/cris/st}}(X) \otimes_{(\cdot)} B,$$

compatibly with the extra structure on both sides. Extracting G_K-invariants will then recover each cohomology theory from the other. $\square$

The remainder of the post constructs the period rings C, B_{HT}, B_{dR}, B_{cris}, and B_{st}, in increasing order of the structure they see.

The completed algebraic closure C

The simplest period ring is $C = \widehat{\overline{K}}$ itself. It is a complete algebraically closed nonarchimedean field carrying a continuous G_K-action. Its usefulness rests on a foundational theorem of Tate (with contributions of Ax and Sen), which computes its Galois invariants.

Theorem (Ax–Sen–Tate). The inclusion $K \hookrightarrow C^{G_K}$ is an equality, $C^{G_K} = K$. More generally, for the Tate twists, $C(n)^{G_K} = 0$ whenever $n \neq 0$.

Proof sketch. The inclusion $K \subseteq C^{G_K}$ is clear. The reverse inclusion is the heart of the matter and rests on the Ax–Sen estimate: there is a constant such that if $x \in C$ is moved only slightly by G_K, meaning $|g(x) - x| \leq \varepsilon$ for all g, then x lies within distance $C \cdot \varepsilon$ of K. Applied to a genuinely invariant $x \in C^{G_K}$ (where ε may be taken arbitrarily small by approximating x by algebraic elements), the estimate forces x to lie in the closure of K inside C, which is K itself since K is complete. We only indicate this; the quantitative approximation lemma is the technical core and is genuinely nontrivial.

For the twisted statement, the key input is Tate's computation of the continuous cohomology of C under $\Gamma = \mathrm{Gal}(K_\infty/K)$, where $K_\infty = K(\mu_{p^\infty})$. Tate's normalized trace maps split C (as a topological Γ-module, up to the completed field $\widehat{K_\infty}$) into an invariant part and a part on which Γ acts with no invariants for a nontrivial power of χ_{cyc}. A nonzero element of $C(n)^{G_K}$ with $n \neq 0$ would produce a line in C on which G_K, and hence Γ, acts by $\chi_{\mathrm{cyc}}^{-n} \neq 1$ while being pointwise fixed, which is impossible. Hence $C(n)^{G_K} = 0$ for $n \neq 0$. A complete account is Tate's theorem on the Galois cohomology of C. □

Thus C already detects Tate twists: an element of a C-line is G_K-fixed exactly when the twist vanishes. This single vanishing statement drives the entire theory of Hodge–Tate weights.

Hodge–Tate representations

Let V be a p-adic representation of G_K. For each integer i, define

$$D^i_{\mathrm{HT}}(V) = (V \otimes_{\mathbb{Q}_p} C(i))^{G_K}.$$

Because $C^{G_K} = K$, each $D^i_{\mathrm{HT}}(V)$ is a K-vector space. There is a natural C-linear comparison map

$$\alpha_{\mathrm{HT},V} : \bigoplus_{i \in \mathbb{Z}} D^i_{\mathrm{HT}}(V) \otimes_K C(-i) \;\longrightarrow\; V \otimes_{\mathbb{Q}_p} C.$$

Construction of the map. An element $d \in D^i_{\mathrm{HT}}(V)$ is definitionally a G_K-invariant vector in $V \otimes C(i)$. Tensoring with $C(-i)$ over C and using the canonical isomorphism $C(i) \otimes_C C(-i) \cong C$ (the twist and its inverse cancel), the image of $d \otimes c$ lands in $V \otimes C(i) \otimes C(-i) \cong V \otimes C$. This defines a C-linear map $D^i_{\mathrm{HT}}(V) \otimes_K C(-i) \to V \otimes C$ for each i; summing over i gives $\alpha_{\mathrm{HT},V}$. The map is G_K-equivariant because d is invariant and the identification $C(i) \otimes C(-i) \cong C$ is equivariant. □

The representation V is Hodge–Tate if $\alpha_{\mathrm{HT},V}$ is an isomorphism. The integers i with $D^i_{\mathrm{HT}}(V) \neq 0$ are the Hodge–Tate weights of V, with multiplicity $\dim_K D^i_{\mathrm{HT}}(V)$.

With this convention, $\mathbb{Q}_p(n)$ has Hodge–Tate weight $-n$.

Proof. We compute directly:

$$D^i_{\mathrm{HT}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} C(i))^{G_K} = C(n+i)^{G_K}.$$

By the Ax–Sen–Tate vanishing, this is 0 unless $n+i=0$. When $i=-n$ we have $C(n+i)=C$, so the invariants are $C^{G_K}=K$, one-dimensional. Hence the unique Hodge–Tate weight is $-n$, with multiplicity 1, and the comparison map is visibly an isomorphism of one-dimensional C-spaces. In particular, $\mathbb{Q}_p(n)$ is Hodge–Tate. □

Theorem (Hodge–Tate comparison). Let X/K be smooth and proper. Then $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)$ is Hodge–Tate, and there is a G_K-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C \;\cong\; \bigoplus_{j=0}^{i} H^{i-j}(X,\Omega^j_{X/K})\otimes_K C(-j).$$

Proof idea. This is a deep comparison theorem, so we only indicate the shape of the argument. One first establishes it for building blocks where the periods are explicit: for $\mathbb{G}_m$ and abelian varieties, the decomposition traces through Kummer theory and the Hodge decomposition of the tangent/cotangent spaces, and for projective space, it follows from the twist computation above.

In general, the theorem is obtained by working on the pro-étale site of X_C with Fontaine's period sheaves. That is, one constructs a filtered period sheaf $\mathcal{O}\mathcal{B}_{\text{dR}}$ whose cohomology computes $H^\bullet_{\text{dR}}$ after suitable twisting, proves a p-adic Poincaré lemma identifying its de Rham complex with a resolution of the constant sheaf, and takes the associated graded. The graded of B_{dR} is $B_{\text{HT}}=\bigoplus_i C(i)$, and the de Rham comparison (below) induces the Hodge–Tate comparison on associated gradeds. The precise argument is due to Fontaine, Messing, Faltings, and in the modern pro-étale form, to Scholze. □

For instance, if $X=\mathbb{P}^1_K$, then $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}},\mathbb{Q}_p)\cong\mathbb{Q}_p(-1)$, and the Hodge–Tate comparison reads

$$\mathbb{Q}_p(-1)\otimes_{\mathbb{Q}_p} C \;\cong\; H^1(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1/K})\otimes_K C(-1).$$

Since $H^1(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1/K})\cong K$ is one-dimensional, this collapses to the tautology $C(-1)\cong C(-1)$, consistent with the single Hodge–Tate weight 1.

The Hodge–Tate period ring

The Hodge–Tate decomposition can be packaged using the graded period ring

$$B_{\text{HT}}=\bigoplus_{i\in\mathbb{Z}} C(i),$$

which is a graded C-algebra with G_K-action (the multiplication uses $C(i)\otimes_C C(j)\cong C(i+j)$). For any p-adic representation V, set

$$D_{\text{HT}}(V)=(V\otimes_{\mathbb{Q}_p} B_{\text{HT}})^{G_K},$$

a graded K-vector space whose degree-i piece is $D^i_{\mathrm{HT}}(V)$. The representation V is Hodge–Tate exactly when $\dim_K D_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V$, the dimension on the left being the total dimension across all graded pieces.

Proof. Since $B_{\mathrm{HT}} = \bigoplus_i C(i)$ and tensoring and invariants commute with finite direct sums,

$$D_{\mathrm{HT}}(V) = \bigoplus_i (V \otimes C(i))^{G_K} = \bigoplus_i D^i_{\mathrm{HT}}(V).$$

The comparison map $\alpha_{\mathrm{HT},V}$ is C-linear between two C-vector spaces of dimensions $\sum_i \dim_K D^i_{\mathrm{HT}}(V)$ and $\dim_C(V \otimes C) = \dim_{\mathbb{Q}_p} V$. It is always injective (the general regularity argument given below for B_{dR} applies verbatim with B_{HT} in place of B_{dR}, using $B_{\mathrm{HT}}^{G_K} = K$ in degree 0). An injective linear map of finite-dimensional spaces is an isomorphism precisely when the dimensions agree. Hence V is Hodge–Tate if and only if $\sum_i \dim_K D^i_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V$. ◻

The ring B_{HT} sees only the associated graded of the Hodge filtration. To recover the full filtration, one needs a period ring that remembers how the graded pieces are glued, namely B_{dR}. Constructing it requires descending to characteristic p and back; we explain this shortly.

The tilt of C

The more refined period rings are built from the tilt of C, its characteristic-p avatar. Define

$$\mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p,$$

the inverse limit of $\mathcal{O}_C/p$ along the p-power (Frobenius) map. An element of $\mathcal{O}_C^\flat$ is a sequence $x = (x_0, x_1, x_2, \ldots)$ with $x_{n+1}^p = x_n$ in $\mathcal{O}_C/p$ for all n. The fraction field $C^\flat = \mathrm{Frac}(\mathcal{O}_C^\flat)$ is a complete algebraically closed nonarchimedean field of characteristic p.

Proof sketch. Addition and multiplication are componentwise, and since each $\mathcal{O}_C/p$ has characteristic p, so does the inverse limit; thus $\mathcal{O}_C^\flat$ is an $\mathbb{F}_p$-algebra. To define its valuation we use the untilting map $\sharp$ constructed just below, setting $|x| = |x^\sharp|$; this is multiplicative, and one checks it is a valuation with the same value group as $\mathcal{O}_C$ (the p-power roots make the group p-divisible). Completeness of $C^\flat$ for this valuation follows from completeness of C, since a Cauchy sequence in $\mathcal{O}_C^\flat$ is a compatible system of Cauchy sequences in $\mathcal{O}_C/p$. Algebraic closedness is the tilting equivalence: finite extensions of $C^\flat$

correspond functorially to finite extensions of C, and C is algebraically closed, so $C^\flat$ has no nontrivial finite extensions. (Here, we treat the tilting equivalence as an external input, as it is the foundational theorem of perfectoid theory.) □

The path back to characteristic 0 is the multiplicative map $(\cdot)^\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_C$. Given $x = (x_0, x_1, \ldots) \in \mathcal{O}_C^\flat$, choose arbitrary lifts $\widetilde{x}_n \in \mathcal{O}_C$ of $x_n \in \mathcal{O}_C/p$, and define

$$x^\sharp = \lim_{n\to\infty} \widetilde{x}_n^{\,p^n}.$$

Convergence and well-definedness. The key elementary fact is the lifting estimate: if $a \equiv b \pmod{p^m}$ in $\mathcal{O}_C$ with $m \geq 1$, then $a^p \equiv b^p \pmod{p^{m+1}}$. Indeed

$$a^p - b^p = (a-b)\sum_{j=0}^{p-1} a^j b^{p-1-j},$$

where $a - b \in p^m\mathcal{O}_C$, and each summand satisfies $a^j b^{p-1-j} \equiv a^{p-1} \pmod{p}$ (since $a \equiv b \pmod{p}$), so the sum is $\equiv p\, a^{p-1} \equiv 0 \pmod{p}$. The product then lies in $p^{m+1}\mathcal{O}_C$.

Now, the compatibility $x_{n+1}^p = x_n$ in $\mathcal{O}_C/p$ means $\widetilde{x}_{n+1}^p \equiv \widetilde{x}_n \pmod{p}$. Raising to the p^n-th power and applying the estimate n times upgrades this to

$$\widetilde{x}_{n+1}^{\,p^{n+1}} = \left(\widetilde{x}_{n+1}^{\,p}\right)^{p^n} \equiv \widetilde{x}_n^{\,p^n} \pmod{p^{n+1}}.$$

Hence, $(\widetilde{x}_n^{\,p^n})_n$ is Cauchy in the p-adically complete ring $\mathcal{O}_C$ and converges. If $\widetilde{x}'_n$ is another choice of lifts, then $\widetilde{x}'_n \equiv \widetilde{x}_n \pmod{p}$, so the same estimate gives $(\widetilde{x}'_n)^{p^n} \equiv \widetilde{x}_n^{\,p^n} \pmod{p^{n+1}}$, and the two limits coincide. Thus, $x^\sharp$ is well defined and independent of lifts. □

The map $x \mapsto x^\sharp$ is multiplicative but not additive.

Proof. Multiplicativity is immediate from $(\widetilde{x}_n\widetilde{y}_n)^{p^n} = \widetilde{x}_n^{\,p^n}\widetilde{y}_n^{\,p^n}$ and passage to the limit, since one may lift xy by the products $\widetilde{x}_n\widetilde{y}_n$. Additivity fails because addition in $\mathcal{O}_C^\flat$ is the limiting operation

$$(x+y)^\sharp = \lim_{n\to\infty}\left(\widetilde{x}_n + \widetilde{y}_n\right)^{p^n},$$

which differs from $x^\sharp + y^\sharp$ in general: the binomial expansion of $(\widetilde{x}_n + \widetilde{y}_n)^{p^n}$ contains cross terms that do not vanish p-adically before the limit is taken. Concretely, over $\mathcal{O}_{\mathbb{C}_p}$ one already sees $(1+1)^\sharp \neq 1^\sharp + 1^\sharp$ for suitable non-integer tilts. □

This construction is the first appearance of the characteristic-p shadow of C. It offers a glimpse of perfectoid tilting and, eventually, of the Fargues–Fontaine curve.

The ring A_{inf}

Define $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$, the ring of p-typical Witt vectors of $\mathcal{O}_C^\flat$. Since $\mathcal{O}_C^\flat$ is perfect (Frobenius is bijective, being the shift on the tilting inverse limit), A_{inf} is p-torsion-free and p-adically complete, and every element has a unique Witt expansion

$$\sum_{n\geq 0} p^n[x_n], \qquad x_n \in \mathcal{O}_C^\flat,$$

where $[x]$ denotes the Teichmüller lift. There is a canonical surjective ring homomorphism $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$, determined by $\theta([x]) = x^\sharp$ and hence on Witt expansions by

$$\theta\Big(\sum_{n\geq 0} p^n[x_n]\Big) = \sum_{n\geq 0} p^n x_n^\sharp.$$

It is instructive to explain why this is a morphism of rings. :::{.proof-like name="Explanation."} The formula is dictated by the universal property of Witt vectors. For a perfect $\mathbb{F}_p$-algebra R, the ring $W(R)$ is the unique p-adically complete, p-torsion-free ring with $W(R)/p \cong R$, and it is initial among such p-adic thickenings. The pair $(\mathcal{O}_C,\ \mathcal{O}_C^\flat \xrightarrow{\sharp} \mathcal{O}_C \bmod p)$ presents $\mathcal{O}_C$ as a p-adic thickening of $\mathcal{O}_C^\flat$ modulo p (indeed $\mathcal{O}_C^\flat/(\text{ker of first projection}) \cong \mathcal{O}_C/p$), so the universal property produces a unique ring homomorphism $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ lifting the identity modulo p. On Teichmüller representatives this map is forced to be $[x] \mapsto x^\sharp$, and additivity together with p-adic continuity extends it to all Witt expansions. The Witt addition and multiplication laws are exactly compatible with this because they are engineered to make $W(-)$ a functor to p-adic thickenings. :::

The kernel $\ker(\theta)$ is a principal ideal, generated by a non-zero-divisor. We fix a generator $\xi \in \ker(\theta)$.

Construction. Choose a compatible system $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_C^\flat$ of p-power roots of p (possible since $\mathcal{O}_C$ is p-divisibly rich and C is algebraically closed), so that $(p^\flat)^\sharp = p$. Then

$$\xi = [p^\flat] - p \in A_{\mathrm{inf}}$$

satisfies $\theta(\xi) = (p^\flat)^\sharp - p = 0$, so $\xi \in \ker\theta$. In the cyclotomic variant one takes $\xi = ([\epsilon]-1)/([\epsilon^{1/p}]-1) = \sum_{j=0}^{p-1}[\epsilon^{j/p}]$ for the system ϵ of p-power roots of unity introduced below; a short computation gives $\theta(\xi) = \sum_{j=0}^{p-1}\zeta_p^j = 0$. That such a ξ generates the whole kernel, and is a non-zero-divisor, is the statement that $A_{\mathrm{inf}} \to \mathcal{O}_C$ realizes $\mathcal{O}_C$ as a perfectoid untilt: the kernel of θ for the untilt of a perfectoid ring is invertible, and over A_{inf} invertible ideals with the relevant reduction are principal. We take this principality as external input, since a self-contained proof requires the deformation theory of perfectoid rings. □

The ring $A_{\inf}$ carries a Frobenius $\varphi : A_{\inf} \to A_{\inf}$ induced by the Witt-vector Frobenius, satisfying $\varphi([x]) = [x^p]$. The group G_K acts on C, hence on $\mathcal{O}_C/p$, hence on $\mathcal{O}_C^\flat$ and on $A_{\inf} = W(\mathcal{O}_C^\flat)$ by functoriality, and the map θ is G_K-equivariant (it is built from the equivariant $\sharp$). Thus, $A_{\inf}$ already carries the three structures that drive the rest of the plot: a G_K-action, a Frobenius φ, and the specialization $\theta : A_{\inf} \to \mathcal{O}_C$.

The de Rham period ring B_{dR}

The ring B_{dR}^+ is defined as the $\ker(\theta)$-adic completion of $A_{\inf}[1/p]$,

$$B_{\mathrm{dR}}^+ = \varprojlim_n A_{\inf}[1/p]/\ker(\theta)^n.$$

The map θ extends to a surjection $\theta : B_{\mathrm{dR}}^+ \to C$, and B_{dR}^+ is a complete discrete valuation ring with residue field C and maximal ideal $\ker(\theta) = (\xi)$; any generator of $\ker(\theta)$ is a uniformizer.

To produce the period of the Tate twist, choose a compatible system of p-power roots of unity, $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_C^\flat$, with ζ_{p^n} primitive. Since $\theta([\epsilon]) = \epsilon^\sharp = \lim_n \zeta_{p^n}^{p^n} = 1$, the Teichmüller element $[\epsilon]$ is a unit in $A_{\inf}$ mapping to $1 \in C$, and $[\epsilon] - 1 \in \ker\theta$. Define

$$t = \log([\epsilon]) = \sum_{n\geq 1} (-1)^{n+1} \frac{([\epsilon]-1)^n}{n}.$$

Convergence of the logarithm. Set $u = [\epsilon] - 1$. Since $\theta(u) = 0$, we have $u \in \ker\theta = (\xi)$, so $u^n \in (\xi)^n$ and $u^n \to 0$ in the $\ker(\theta)$-adic topology of B_{dR}^+. The denominators n are units in $A_{\inf}[1/p]$ (we have inverted p), and their p-adic valuations grow only logarithmically in n, so $u^n/n \to 0$ as well. Because B_{dR}^+ is $\ker(\theta)$-adically complete, the series converges. Thus $t \in B_{\mathrm{dR}}^+$, and in fact t generates $\ker\theta$: modulo $(\xi)^2$ one has $t \equiv u$, and $u = [\epsilon] - 1$ is a uniformizer, so t is too. □

The element t transforms by the cyclotomic character.

Proof. Since $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{\mathrm{cyc}}(g)}$ compatibly, we have $g(\epsilon) = \epsilon^{\chi_{\mathrm{cyc}}(g)}$ in $\mathcal{O}_C^\flat$, where the exponent $\chi_{\mathrm{cyc}}(g) \in \mathbb{Z}_p^\times$ acts through the $\mathbb{Z}_p$-module structure of the p-divisible group μ_{p^∞}. Applying the Teichmüller lift and continuity, $g([\epsilon]) = [\epsilon]^{\chi_{\mathrm{cyc}}(g)}$. Therefore

$$g(t) = g\big(\log[\epsilon]\big) = \log\big(g[\epsilon]\big) = \log\big([\epsilon]^{\chi_{\mathrm{cyc}}(g)}\big) = \chi_{\mathrm{cyc}}(g)\log[\epsilon] = \chi_{\mathrm{cyc}}(g)\, t,$$

where the identity $\log(z^a) = a \log z$ holds for $a \in \mathbb{Z}_p$ by continuity from the case $a \in \mathbb{Z}$. □

Thus, t is the p-adic analogue of $2\pi i$: it spans a G_K-stable line on which the group acts by χ_{cyc}, exactly as $2\pi i$ spans the period of $H^1(\mathbb{G}_m)$. Now define

$$B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t].$$

This is a complete discretely valued field with a decreasing, exhaustive, separated filtration $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$, and its associated graded pieces are Tate twists of C.

Proof. Multiplication by t^i is an isomorphism $B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ \xrightarrow{\sim} t^i B_{\mathrm{dR}}^+/t^{i+1}B_{\mathrm{dR}}^+ = \mathrm{Fil}^i / \mathrm{Fil}^{i+1}$. Since t generates the maximal ideal $\ker\theta$ of the DVR B_{dR}^+, the quotient $B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+$ is the residue field C, with its natural G_K-action. The isomorphism above is not G_K-equivariant on the nose: it multiplies by t^i, and $g(t^i) = \chi_{\mathrm{cyc}}(g)^i t^i$, so it intertwines the action on C with the action twisted by χ_{cyc}^i. Therefore, as a G_K-module, $\mathrm{gr}^i B_{\mathrm{dR}} \cong C(i)$. □

Consequently $\mathrm{gr}^\bullet B_{\mathrm{dR}} \cong \bigoplus_i C(i) = B_{\mathrm{HT}}$. This gives a relationship between the de Rham and Hodge–Tate period rings.

We show that the ring B_{dR} has a continuous G_K-action with $B_{\mathrm{dR}}^{G_K} = K$.

Proof. The inclusion $K \subseteq B_{\mathrm{dR}}^{G_K}$ is clear. Conversely, let $x \in B_{\mathrm{dR}}^{G_K}$ be nonzero, and let i be its valuation, so $x \in \mathrm{Fil}^i \setminus \mathrm{Fil}^{i+1}$. The image $\overline{x}$ of x in $\mathrm{gr}^i B_{\mathrm{dR}} \cong C(i)$ is a nonzero G_K-invariant element, hence lies in $C(i)^{G_K}$. By Ax–Sen–Tate, $C(i)^{G_K} = 0$ for $i \neq 0$ and $C^{G_K} = K$ for $i = 0$; since $\overline{x} \neq 0$, we must have $i = 0$ and $\overline{x} \in K^\times \subseteq C$. Subtracting a lift in K of $\overline{x}$ (using $K \hookrightarrow B_{\mathrm{dR}}^+$, which exists because B_{dR}^+ contains $\overline{K}$ hence K, and its residue map to C restricts to the inclusion $K \hookrightarrow C$) strictly raises the valuation of the invariant element $x - (\mathrm{lift})$. Iterating and using completeness of B_{dR}^+ shows the successive corrections converge, expressing x as an element of K. Hence $B_{\mathrm{dR}}^{G_K} = K$. □

De Rham representations

Let V be a p-adic representation of G_K. Define

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}.$$

Because $B_{\mathrm{dR}}^{G_K} = K$, this is a K-vector space. It inherits a decreasing filtration from B_{dR},

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \cap \left(V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^i B_{\mathrm{dR}}\right).$$

The representation V is de Rham if $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$.

The following bound holds for every V, and being de Rham means it is attained.

Lemma (regularity bound). For every p-adic representation V, one has $\dim_K D_{\mathrm{dR}}(V) \le \dim_{\mathbb{Q}_p} V$.

Proof. It suffices to show that the natural B_{dR}-linear map

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

is injective; comparing B_{dR}-dimensions then gives $\dim_K D_{\mathrm{dR}}(V) \le \dim_{B_{\mathrm{dR}}}(V \otimes B_{\mathrm{dR}}) = \dim_{\mathbb{Q}_p} V$. Equivalently, we show that any K-linearly independent family $d_1, \dots, d_r \in D_{\mathrm{dR}}(V)$ remains B_{dR}-linearly independent in $V \otimes B_{\mathrm{dR}}$.

Suppose not, and take a nontrivial dependence relation $\sum_{i=1}^{s} b_i d_i = 0$ with $b_i \in B_{\mathrm{dR}}$ of minimal length s among all such relations (reindexing so the involved indices are $1, \dots, s$). Since B_{dR} is a field we may normalize $b_1 = 1$. Apply any $g \in G_K$. Because each $d_i \in D_{\mathrm{dR}}(V) = (V \otimes B_{\mathrm{dR}})^{G_K}$ is invariant, g acts only on the coefficients:

$$0 = g\Big(\sum_i b_i d_i\Big) = \sum_i g(b_i)\, d_i.$$

Subtracting the original relation and using $g(b_1) = g(1) = 1 = b_1$ kills the first term:

$$\sum_{i=2}^{s} \big(g(b_i) - b_i\big) d_i = 0.$$

This is a relation of length $< s$, so by minimality it is trivial, giving $g(b_i) = b_i$ for all i and all g. Hence $b_i \in B_{\mathrm{dR}}^{G_K} = K$. But then $\sum_i b_i d_i = 0$ is a nontrivial K-linear relation among $d_1, \dots, d_s$, contradicting their K-linear independence. Therefore no such B_{dR}-relation exists, the map is injective, and the bound follows. □

The same argument, replacing $B_{\mathrm{dR}}^{G_K} = K$ by $B_{\mathrm{cris}}^{G_K} = B_{\mathrm{st}}^{G_K} = K_0$, gives the analogous bounds for D_{cris} and D_{st} below; this is the general phenomenon that B_{dR}, B_{cris}, and B_{st} are $(\mathbb{Q}_p, G_K)$-regular period rings.

Theorem (de Rham comparison). Let X/K be smooth and proper, and set $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Then V is de Rham, and there is a canonical filtered isomorphism $D_{\mathrm{dR}}(V) \cong H^i_{\mathrm{dR}}(X/K)$. Equivalently, there is a canonical G_K-equivariant, filtration-compatible isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \;\cong\; H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}.$$

Proof idea. We merely indicate the mechanism; the full theorem is one of the central results of the subject. On the pro-étale site of X_C, Fontaine's filtered period sheaf $\mathcal{O}\mathcal{B}_{\mathrm{dR}}$ interpolates between the constant sheaf $\mathbb{Q}_p$ and the de Rham complex. A p-adic Poincaré lemma states that the de Rham complex of $\mathcal{O}\mathcal{B}_{\mathrm{dR}}$ (with its connection) resolves B_{dR}, so the pro-étale cohomology of B_{dR} is computed by $H^\bullet_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}$. Comparing with the étale side, where $R\Gamma_{\text{pro-ét}}(X_C, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}$ recovers $H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}$, yields the comparison isomorphism, filtered because $\mathcal{O}\mathcal{B}_{\mathrm{dR}}$ is filtered. Properness gives finiteness and Poincaré duality, while smoothness gives the Poincaré lemma. Taking G_K-invariants and using $B_{\mathrm{dR}}^{G_K} = K$ recovers $D_{\mathrm{dR}}(V) \cong H^i_{\mathrm{dR}}(X/K)$. This is due to Faltings, with the pro-étale formulation due to Scholze. □

Passing to associated gradeds in the de Rham comparison and using $\mathrm{gr}^\bullet B_{\mathrm{dR}} \cong B_{\mathrm{HT}}$ recovers the Hodge–Tate comparison. In particular, every de Rham representation is Hodge–Tate.

The crystalline period ring B_{cris}

The ring B_{dR} sees the filtration but forgets the Frobenius, because the completion that defines it destroys the Frobenius on A_{inf} (the ideal $\ker\theta$ is not Frobenius-stable). To retain the Frobenius, one completes more gently, using divided powers.

Starting from $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$, let A_{cris} be the p-adic completion of the divided-power envelope of A_{inf} with respect to $\ker(\theta)$. Let us explain what this means.

A divided-power structure on an ideal $I \subseteq R$ is a family of maps $\gamma_n : I \to R$ for $n \geq 1$ behaving formally like $\gamma_n(x) = x^n/n!$, meaning $\gamma_n(x) \cdot n! = x^n$, $\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x)$, and the expected addition and scalar rules, even when $n!$ is not invertible in R.

The divided-power envelope $D_{\ker\theta}(A_{\mathrm{inf}})$ is the universal A_{inf}-algebra in which $\ker(\theta)$ acquires a divided-power structure. Adjoining divided powers is exactly what crystalline cohomology requires, since crystalline thickenings are pro-nilpotent thickenings equipped with divided powers on their defining ideals. Define

$$B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p], \qquad B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t].$$

Here, $t = \log[\epsilon]$ already lies in A_{cris}, since the divided powers make the terms $([\epsilon]-1)^n/n = (n-1)!\,\gamma_n([\epsilon]-1)$ converge p-adically. The ring B_{cris} carries a continuous G_K-action, a Frobenius $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$, and a G_K-equivariant embedding $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$.

Construction. The G_K-action and the Frobenius φ on A_{inf} both preserve $\ker(\theta)$ as a set closed under the ambient operations needed to extend divided powers (φ preserves $\ker\theta$ because $\theta \circ \varphi$ and θ agree modulo p on Teichmüller elements up to the divided-power corrections), so both extend to the divided-power envelope A_{cris} and then to B_{cris} after inverting p and t. For the embedding, the universal property of the divided-power envelope maps A_{cris} into any $\ker(\theta)$-adically complete algebra where $\ker\theta$ has divided powers; B_{dR}^{+} qualifies, since $\ker\theta = (\xi)$ becomes topologically nilpotent and the denominators $n!$ are invertible after inverting p. Inverting t on both sides gives $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$; injectivity holds because B_{cris} is a domain mapping nontrivially into the field B_{dR}. □

The Frobenius scales t by p.

Proof. Using $\varphi([\epsilon]) = [\epsilon^p] = [\epsilon]^p$ and continuity of log,

$$\varphi(t) = \varphi\big(\log[\epsilon]\big) = \log\big([\epsilon]^p\big) = p\log[\epsilon] = p\,t.$$

□

Finally, the crystalline invariants are the maximal unramified subfield.

Proof sketch. The invariants are contained in $B_{\mathrm{dR}}^{G_K} = K$, so $B_{\mathrm{cris}}^{G_K} \subseteq K$. The refinement to K_0 comes from Frobenius: an element of $B_{\mathrm{cris}}^{G_K}$ carries a well-defined action of φ, and one shows that the G_K-invariants of B_{cris} are precisely the φ-relevant unramified periods, whose fixed field is the maximal unramified subextension $K_0 = W(k)[1/p]$ rather than all of K. The ramified part of K does not survive because it is not visible to the Frobenius-compatible crystalline structure; only K_0, which is $W(k)[1/p]$ and on which σ acts, is retained. A rigorous proof analyzes the fundamental exact sequences relating B_{cris}, A_{cris}, and C together with Ax–Sen–Tate, and is part of Fontaine's foundational study of B_{cris}. □

Crystalline representations

Let V be a p-adic representation of G_K, and define

$$D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}.$$

This is a K_0-vector space, and the Frobenius on B_{cris} induces a σ-semilinear Frobenius $\varphi : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$ (semilinear because φ acts nontrivially on the coefficient field K_0 through σ).

The representation V is crystalline if $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$.

As in the de Rham case, one always has $\dim_{K_0} D_{\mathrm{cris}}(V) \le \dim_{\mathbb{Q}_p} V$: the regularity argument above applies verbatim with B_{cris} and $K_0 = B_{\mathrm{cris}}^{G_K}$ in place of B_{dR} and K, so any K_0-linearly independent family in $D_{\mathrm{cris}}(V)$ stays B_{cris}-linearly independent in $V \otimes B_{\mathrm{cris}}$.

The embedding $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ makes crystalline representations de Rham, and it identifies their invariants after extending scalars.

Proof that crystalline is de Rham. If V is crystalline, then the comparison map is an isomorphism at the crystalline level,

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \xrightarrow{\ \sim\ } V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

(injective by regularity, surjective because of equality of dimensions). Tensoring along $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ gives

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{dR}} \;\cong\; V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

Taking G_K-invariants and using $B_{\mathrm{dR}}^{G_K} = K$ together with $D_{\mathrm{cris}}(V)^{G_K} = D_{\mathrm{cris}}(V)$ (the K_0-structure is already G_K-fixed) yields

$$D_{\mathrm{dR}}(V) \cong D_{\mathrm{cris}}(V) \otimes_{K_0} K,$$

which has K-dimension $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$. Hence V is de Rham, and $D_{\mathrm{dR}}(V)$ acquires its filtration through the embedding $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$, with the Frobenius on $D_{\mathrm{cris}}(V)$ as extra structure that D_{dR} alone does not record. □

Theorem (crystalline comparison). Let X/K be smooth and proper with good reduction, let $\mathcal{X}/\mathcal{O}_K$ be a smooth proper model, and let $X_k = \mathcal{X} \times_{\mathcal{O}_K} k$ be the special fiber. Then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, and there is a canonical Frobenius-compatible isomorphism

$$D_{\mathrm{cris}}(V) \;\cong\; H^i_{\mathrm{cris}}(X_k/W(k))[1/p].$$

Equivalently,

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \;\cong\; H^i_{\mathrm{cris}}(X_k/W(k))[1/p] \otimes_{K_0} B_{\mathrm{cris}}.$$

Proof idea. The special fiber X_k has crystalline cohomology over $W(k)$, a finitely generated $W(k)$-module with a σ-semilinear Frobenius; inverting p gives a φ-module over K_0. The generic fiber has p-adic étale cohomology with G_K-action. Good reduction means the smooth proper model $\mathcal{X}$ connects them with no monodromy. The comparison is realized by a period morphism built from the crystalline nature of B_{cris}: the divided-power structure on $\ker\theta$ is exactly what permits crystalline thickenings of X_k over A_{cris}/p^m to communicate with the characteristic-zero étale cohomology. Now, compatibility with Frobenius is automatic, because φ is present on both B_{cris} and H^i_{cris}, and the filtration is recovered after passing to B_{dR}. This is the Fontaine conjecture C_{cris}, proved by Fontaine–Messing, Faltings, and others. □

Thus, good reduction is reflected representation-theoretically by the crystalline property, i.e., the p-adic representation remembers the crystalline cohomology of the smooth special fiber, together with its Frobenius.

The semistable period ring B_{st}

Good reduction is a stringent condition. Many varieties have only semistable reduction, where the special fiber is a normal-crossings degeneration, whereupon Frobenius alone underdetermines the representation (as one must also remember the monodromy of the degeneration). The corresponding period ring is this instance is B_{st}. It contains B_{cris} and carries an additional monodromy operator $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, together with the Frobenius φ, satisfying the commutation relation

$$N\varphi = p\,\varphi N.$$

One realizes B_{st} by adjoining a single logarithmic period. Fix a compatible system $p^\flat = (p, p^{1/p}, \dots) \in \mathcal{O}_C^\flat$ of p-power roots of p, and adjoin a formal variable $\ell = \log[p^\flat]$ (its image in B_{dR} depends on a choice of branch of $\log p$, and different choices differ by elements of B_{cris}). Specifically, one may write

$$B_{\mathrm{st}} \cong B_{\mathrm{cris}}[\ell],$$

a polynomial ring in ℓ over B_{cris}, with the Frobenius and monodromy determined by

$$\varphi(\ell) = p\,\ell, \qquad N = -\frac{d}{d\ell}.$$

The construction is independent of the auxiliary choices up to canonical isomorphism.

Verification. It suffices to check the relation on B_{cris} and on powers ℓ^m, since both sides are additive and $B_{\mathrm{st}} = \bigoplus_m B_{\mathrm{cris}}\,\ell^m$. On B_{cris} the operator N vanishes (it is $-d/d\ell$ and B_{cris} is ℓ-independent), so both $N\varphi$ and $p\varphi N$ vanish there. On ℓ^m with $m \geq 1$,

$$N\varphi(\ell^m) = N\big(p^m\ell^m\big) = p^m \cdot (-m\ell^{m-1}) = -m\,p^m\ell^{m-1},$$

while

$$p\,\varphi N(\ell^m) = p\,\varphi\big(-m\ell^{m-1}\big) = p \cdot (-m)\,p^{m-1}\ell^{m-1} = -m\,p^m\ell^{m-1}.$$

The two agree, so $N\varphi = p\varphi N$ on all of B_{st}. Interpreting ℓ as a logarithm, the relation is the algebraic manifestation of the fact that Frobenius rescales logarithmic monodromy by p: if $\varphi(\ell) = p\ell$, then differentiating after applying φ picks up an extra factor of p. $\square$

The ring B_{st} thus carries rich structure by way of a G_K-action, a Frobenius φ, a monodromy operator N, and a G_K-equivariant embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ (depending on the choice of branch of $\log p$). Its invariants are again $B_{\mathrm{st}}^{G_K} = K_0$, since G_K acts on ℓ through $g(\ell) = \ell + \chi_{\mathrm{cyc}}$-multiple of t, which contributes nothing new to the invariants beyond $B_{\mathrm{cris}}^{G_K} = K_0$.

Semistable representations

For a p-adic representation V, define

$$D_{\mathrm{st}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K}.$$

This is a K_0-vector space equipped with a σ-semilinear Frobenius φ and a K_0-linear monodromy operator N, inherited from B_{st} and satisfying $N\varphi = p\varphi N$.

The representation V is semistable if $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$.

As before, the regularity bound gives $\dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V$ for every V. The chain of embeddings $B_{\mathrm{cris}} \subseteq B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$ furnishes the following hierarchy of representation classes.

Proposition (the hierarchy). For every p-adic representation,

$$\text{crystalline} \implies \text{semistable} \implies \text{de Rham} \implies \text{Hodge–Tate.}$$

Proof. Suppose V is crystalline, so $D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$. Tensoring along $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{st}}$ gives $D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$; taking invariants shows $D_{\mathrm{st}}(V) \supseteq D_{\mathrm{cris}}(V)$ has full dimension, so V is semistable (indeed $D_{\mathrm{st}}(V) = D_{\mathrm{cris}}(V)$ with $N = 0$ in this case).

Suppose V is semistable, so $D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$. Tensoring along $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ and taking G_K-invariants (using $B_{\mathrm{dR}}^{G_K} = K$) gives $D_{\mathrm{dR}}(V) \cong D_{\mathrm{st}}(V) \otimes_{K_0} K$ of full K-dimension, so V is de Rham.

Suppose V is de Rham, so $D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$. Passing to the associated graded with respect to the t-adic filtration and using $\mathrm{gr}^{\bullet} B_{\mathrm{dR}} \cong B_{\mathrm{HT}}$ produces the Hodge–Tate comparison isomorphism $\mathrm{gr}\, D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{HT}} \cong V \otimes B_{\mathrm{HT}}$, so V is Hodge–Tate. This gives the full chain of implications. □

The hierarchy is strict: indeed, there are Hodge–Tate representations that are not de Rham, de Rham representations that are not semistable, and semistable representations that are not crystalline, the last illustrated by elliptic curves with multiplicative reduction below.

Theorem (semistable comparison). Let X/K be smooth and proper with semistable reduction. Then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is semistable, and there is a canonical comparison isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \;\cong\; H^i_{\text{log-cris}}(X_k)[1/p] \otimes_{K_0} B_{\mathrm{st}},$$

compatible with G_K-action, Frobenius, monodromy, and (after extension to B_{dR}) the filtration.

Proof idea. Under semistable reduction the special fiber is a reduced normal-crossings divisor, which is smooth in the sense of logarithmic geometry once equipped with its natural log structure. Log-crystalline (Hyodo–Kato) cohomology replaces crystalline cohomology, and the log structure produces the monodromy operator N, measuring the combinatorics of how the components of X_k meet. The semistable comparison identifies the generic-fiber étale cohomology with the log-crystalline cohomology of the special fiber after extending scalars to B_{st}; the extra logarithmic period ℓ is exactly what encodes the monodromy. This is the conjecture C_{st}, proved by Kato, Tsuji, and others. □

Thus, semistable reduction is reflected representation-theoretically by the appearance of a nonzero monodromy operator.

A table of period rings

The period rings introduced above may be summarized as follows.

Period ring	Extra structure	Detects
C	G_K-action	Hodge–Tate weights
B_{HT}	$\mathbb{Z}$-grading	Hodge–Tate decomposition
B_{dR}	filtration	de Rham representations
B_{cris}	$\varphi,\ B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$	crystalline representations
B_{st}	$\varphi,\ N,\ B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$	semistable representations

The corresponding Fontaine functors are the invariants $D_\star(V) = (V \otimes_{\mathbb{Q}_p} B_\star)^{G_K}$ for $\star \in \{\mathrm{HT}, \mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$, and the hierarchy of representation classes is

$$\text{crystalline} \implies \text{semistable} \implies \text{de Rham} \implies \text{Hodge–Tate.}$$

Example: Tate twists

Let $V = \mathbb{Q}_p(n)$, with basis e_n satisfying $g(e_n) = \chi_{\mathrm{cyc}}(g)^n e_n$. Recall $g(t) = \chi_{\mathrm{cyc}}(g)t$. Then $e_n \otimes t^{-n}$ is G_K-invariant in $\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$.

Proof. We compute directly, using multiplicativity of the actions on the two tensor factors:

$$g(e_n \otimes t^{-n}) = g(e_n) \otimes g(t)^{-n} = \chi_{\mathrm{cyc}}(g)^n e_n \otimes \big(\chi_{\mathrm{cyc}}(g)t\big)^{-n} = \chi_{\mathrm{cyc}}(g)^n \chi_{\mathrm{cyc}}(g)^{-n} (e$$

Hence, $e_n \otimes t^{-n}$ is invariant. Since $\mathbb{Q}_p(n)$ is one-dimensional and t is a unit in B_{dR}, this single invariant spans, so $D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = K \cdot (e_n \otimes t^{-n})$ has full dimension 1, and $\mathbb{Q}_p(n)$ is de Rham. □

The same invariant works over B_{cris} (as $t \in B_{\mathrm{cris}}^\times$), giving $D_{\mathrm{cris}}(\mathbb{Q}_p(n)) = K_0 \cdot (e_n \otimes t^{-n})$, so $\mathbb{Q}_p(n)$ is crystalline. Since $\varphi(t) = pt$,

$$\varphi(e_n \otimes t^{-n}) = e_n \otimes \varphi(t)^{-n} = e_n \otimes (pt)^{-n} = p^{-n}(e_n \otimes t^{-n}),$$

so Frobenius acts on the crystalline line by p^{-n}. The Hodge–Tate weight of $\mathbb{Q}_p(n)$ is $-n$, as computed earlier. In particular $\mathbb{Q}_p(-1)$ is crystalline with Hodge–Tate weight 1 and crystalline Frobenius eigenvalue p, consistent with the geometric identification $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-1)$.

Example: projective space

Let $X = \mathbb{P}^n_K$. Then $H^{2i}_{\text{ét}}(\mathbb{P}^n_{\overline{K}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-i)$ for $0 \leq i \leq n$, and the odd cohomology vanishes, as established above. On the de Rham side, $H^{2i}_{\text{dR}}(\mathbb{P}^n/K)$ is one-dimensional over K, generated by the i-th power of the hyperplane class, and the Hodge filtration places this generator in degree i (it lives in Fil^i but not Fil^{i+1}).

The standard model $\mathbb{P}^n_{\mathcal{O}_K}$ is smooth and proper, so $\mathbb{P}^n_K$ has good reduction, and crystalline comparison applies:

$$D_{\text{cris}}(\mathbb{Q}_p(-i)) \cong H^{2i}_{\text{cris}}(\mathbb{P}^n_k/W(k))[1/p],$$

a K_0-line on which Frobenius acts by p^i (matching the computation for $\mathbb{Q}_p(-i)$ above). Thus, the single Tate twist $\mathbb{Q}_p(-i)$ simultaneously records the Hodge–Tate weight i, the de Rham filtration jump in degree i, and the crystalline Frobenius slope i. This coincidence of three integers is the dictionary of p-adic Hodge theory in its simplest incarnation: for a Tate twist, the Hodge–Tate weight, the filtration jump, and the Frobenius slope all agree.

Example: elliptic curves

Let E/K be an elliptic curve. Its p-adic Tate module is $T_p(E) = \varprojlim_n E[p^n](\overline{K})$, and $V_p(E) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a two-dimensional p-adic representation of G_K.

Proof. For each n, the p^n-torsion $E[p^n](\overline{K})$ is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank 2 (an elliptic curve over an algebraically closed field of characteristic $\neq p$ has $E[m] \cong (\mathbb{Z}/m)^2$). The transition maps $E[p^{n+1}] \to E[p^n]$ are multiplication by p, which are surjective with the expected kernels, so the inverse limit $T_p(E)$ is a free $\mathbb{Z}_p$-module of rank 2. The Galois action on the coordinates of torsion points is continuous (each $E[p^n]$ is defined over a finite extension), giving a continuous G_K-action on $T_p(E)$, and inverting p produces the two-dimensional $\mathbb{Q}_p$-representation $V_p(E)$. □

Étale cohomology recovers this, up to duality:

$$H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p) \cong V_p(E)^\vee.$$

Proof. For a smooth proper curve, $H^1_{\text{ét}}$ is canonically dual to the rational Tate module of the Jacobian: $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p) \cong V_p(\operatorname{Jac} E)^\vee$. For an elliptic curve the Jacobian is E itself (via the canonical principal polarization sending E to $\operatorname{Pic}^0 E$), so $\operatorname{Jac} E \cong E$ and $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p) \cong V_p(E)^\vee$. □

The Hodge–Tate comparison in degree 1 reads

$$H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \;\cong\; H^1(E, \mathcal{O}_E) \otimes_K C \;\oplus\; H^0(E, \Omega^1_{E/K}) \otimes_K C(-1),$$

and both K-spaces on the right are one-dimensional.

Proof. An elliptic curve has genus 1, so $\dim_K H^0(E, \Omega^1_{E/K}) = 1$ (the space of invariant differentials). Serre duality on the curve gives $H^1(E, \mathcal{O}_E)^\vee \cong H^0(E, \Omega^1_{E/K})$, whence $\dim_K H^1(E, \mathcal{O}_E) = 1$ as well. □

Therefore, $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p)$ has Hodge–Tate weights 0 and 1, each with multiplicity 1. The reduction type of E refines this into the finer structure detected by B_{cris} and B_{st}.

If E has good reduction, E extends to a smooth proper elliptic scheme over $\mathcal{O}_K$, so crystalline comparison applies and $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, with D_{cris} identified with the crystalline cohomology of the special fiber and its Frobenius (whose characteristic polynomial encodes the point count of E mod π).

If E has multiplicative (semistable, not good) reduction, then $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p)$ is semistable but not crystalline, and its monodromy operator N is nonzero.

Proof. Multiplicative reduction is modeled analytically by a Tate curve $E_q = \mathbb{G}_m^{\text{an}}/q^{\mathbb{Z}}$ for a parameter $q \in K^\times$ with $0 < |q| < 1$. The rigid-analytic uniformization $\mathbb{G}_m^{\text{an}} \to E_q$ realizes $V_p(E_q)$ as an extension

$$0 \to \mathbb{Q}_p(1) \to V_p(E_q) \to \mathbb{Q}_p \to 0,$$

where the sub $\mathbb{Q}_p(1)$ comes from the roots of unity inside $\mathbb{G}_m$ and the quotient $\mathbb{Q}_p$ from the lattice $q^{\mathbb{Z}}$. This extension is nonsplit, and its class in $\operatorname{Ext}^1_{G_K}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong K^\times \widehat{\otimes} \mathbb{Q}_p$ is exactly q (Kummer theory). Because the degeneration is logarithmic, the representation is semistable, and the nonsplitting corresponds precisely to a nonzero monodromy operator N on D_{st}, which maps the weight-0 line onto the weight-1 line and records the loop created by the nodal special fiber. Since $N \neq 0$, the representation is not crystalline, as crystalline representations have $N = 0$. □

As we hinted at above, elliptic curves thus give the first legitimately nontrivial instance of the hierarchy: good reduction produces crystalline representations, while multiplicative reduction produces semistable representations that are not crystalline, distinguished exactly by the vanishing or nonvanishing of N.

The geometric dictionary

The comparison theorems assemble into a dictionary between the geometry of X/K and the linear-algebraic output extracted from $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which we summarize in the following table.

Geometry of X/K	Type of $H^i_{\text{ét}}$	Linear-algebra output
smooth proper	de Rham	filtered K-vector space
good reduction	crystalline	φ-module with filtration
semistable reduction	semistable	(φ, N)-module with filtration

The functors D_{dR}, D_{cris}, and D_{st} identify the period-ring invariants with actual geometric cohomology:

$$D_{\text{dR}}\big(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \cong H^i_{\text{dR}}(X/K),$$

and, under good reduction,

$$D_{\text{cris}}\big(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \cong H^i_{\text{cris}}(X_k/W(k))[1/p].$$

The upshot is that a topological Galois representation has been resolved into filtered linear algebra over K and K_0, equipped with a Frobenius and a monodromy operator.

This is the entry point to the next post, which reverses the direction of inquiry. Rather than starting from a representation and extracting its invariants, one starts from the linear algebra, the filtered (φ, N)-modules over K_0, and asks which of them actually arise from representations. The answer is Fontaine's theorem that the essential image consists of the weakly admissible modules, and pursuing it leads to Hodge and Newton polygons, weak admissibility, and eventually the Fargues–Fontaine curve.