Trace formulas, old and new

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The goal of this post is to give an exposition of trace formulas, which are indispensible tools across several areas of mathematics. They feature perhaps most prominently in the theory of automorphic forms, but before explaining how, we emphasize that the trace formula is at heart, a method that rests on more primitive scaffolding.

The recurring motif is that we construct an operator, compute its trace in two different ways, and thereby identify two kinds of information that are not visibly related. It begins with the finite-dimensional prototype, where the trace of an operator with kernel $K(x,y)$ is the sum of the diagonal values $K(x,x)$. When the operator comes from a map of a finite set, this diagonal sum counts fixed points; when it comes from a finite group action, the same mechanism gives the finite group trace formula and Burnside's lemma. The Grothendieck-Lefschetz trace formula then augments this principle from finite sets to varieties over finite fields: here, Frobenius-fixed points are rational points, and they are counted by the alternating trace of Frobenius on compactly supported étale cohomology.

The second half explains why the same idea becomes so central in the theory of automorphic forms. A compact quotient $\Gamma \setminus G$ replaces the finite set, and a test function $f \in C_c^\infty(G)$ defines a convolution operator $R(f)$ on $L^2(\Gamma \setminus G)$. Computing $\mathrm{Tr}(R(f))$ spectrally gives a sum over irreducible representations occurring in $L^2(\Gamma \setminus G)$; computing it geometrically requires writing down the kernel, restricting to the diagonal, and unfolding the resulting integral into a sum over conjugacy classes and orbital integrals. This yields the compact automorphic trace formula. In the special case of compact hyperbolic surfaces, it becomes the Selberg trace formula, relating eigenvalues of the Laplacian to lengths of closed geodesics. The final sections explain how this compact picture points toward the full adelic and noncompact theory, where Arthur's trace formula introduces truncation, weighted orbital integrals, Eisenstein series, and the comparison of trace formulae used in Langlands transfer.

The prototype

Let X be a finite set, and let $V = \mathrm{Fun}(X, \mathbb{C})$ be the vector space of complex-valued functions on X. For each $x \in X$, let δ_x be the delta function such that $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for all $y \neq x$. Then, $\{\delta_x : x \in X\}$ is a basis of V.

A function $K : X \times X \to \mathbb{C}$ defines a linear operator $T_K : V \to V$ given by

$$(T_K f)(x) = \sum_{y \in X} K(x, y) f(y).$$

That is, $K(x, y)$ is the matrix coefficient of T_K in row x, column y. Now, the map $K \mapsto T_K$ induces an isomorphism of vector spaces

$$\operatorname{Fun}(X \times X, \mathbb{C}) \cong \operatorname{End}_{\mathbb{C}}(\operatorname{Fun}(X, \mathbb{C})).$$

Indeed, since X is finite (say $\# X = n$), both vector spaces have dimension n^2. We show first that the map is injective. Suppose $T_K = 0$. Then, for every $y \in X$,

$$0 = T_K(\delta_y).$$

Evaluating at x, we get

$$0 = (T_K \delta_y)(x) = \sum_{z \in X} K(x, z) \delta_y(z) = K(x, y).$$

Thus, $K(x, y) = 0$ for all x, y, so $K = 0$. Hence, the map is injective, and since the dimensions agree, it is an isomorphism. ☐

Now, we prove the most basic form of the trace formula.

Theorem (Diagonal trace formula). For every kernel $K : X \times X \to \mathbb{C}$,

$$\operatorname{Tr}(T_K) = \sum_{x \in X} K(x, x).$$

Proof The matrix of T_K in the basis $\{\delta_x\}$ has entries

$$(T_K)_{x,y} = K(x, y).$$

The trace of a finite matrix is the sum of its diagonal entries. Therefore,

$$\operatorname{Tr}(T_K) = \sum_{x \in X} (T_K)_{x,x} = \sum_{x \in X} K(x, x).$$

☐

Fixed points as traces

Now, let $\sigma : X \to X$ be any map of finite sets. It induces on VV a pullback operator

$$\sigma^* : V \to V$$

defined by

$$(\sigma^* f)(x) = f(\sigma(x)).$$

A foundational observation is that $\operatorname{Tr}(\sigma^*) = \# \operatorname{Fix}(\sigma)$. Indeed, we compute the matrix of σ^* in the delta basis. For $y \in X$,

$$\sigma^*(\delta_y)(x) = \delta_y(\sigma(x)).$$

Thus, $\sigma^*(\delta_y)(x) = 1$ if and only if $\sigma(x) = y$. Therefore,

$$\sigma^*(\delta_y) = \sum_{x \in X, \sigma(x) = y} \delta_x.$$

So, the coefficient of δ_x in $\sigma^*(\delta_y)$ is 1 if $\sigma(x) = y$, and 0 otherwise. The diagonal coefficient corresponding to $x = y$ is therefore 1 precisely when

$$\sigma(x) = x.$$

The result follows.

This is the finite-set Lefschetz trace formula, and the key thrust is that fixed points of a map correspond to the trace of the induced operator on functions.

Finite group trace formula

Let G be a finite group acting on a finite set X. Again, let $V = \mathrm{Fun}(X, \mathbb{C})$. The action of G on X gives rise to a representation

$$\rho : G \to \mathrm{GL}(V)$$

via

$$(\rho(g)f)(x) = f(g^{-1}x).$$

Let $X^g = \{x \in X : gx = x\}$. Then, we note that $\mathrm{Tr}(\rho(g)) = \#X^g$, furthering the paradigm described at the end of the previous section. To this end, the operator $\rho(g)$ acts on the delta basis as follows: for $x, y \in X$,

$$(\rho(g)\delta_y)(x) = \delta_y(g^{-1}x).$$

This equals 1 exactly when $g^{-1}x = y$, i.e. $x = gy$. Thus,

$$\rho(g)\delta_y = \delta_{gy}.$$

In words, the matrix of $\rho(x)$ is the permutation matrix corresponding to the permutation $x \mapsto gy$ of X.

A permutation matrix has diagonal entry 1 in position x exactly when $gx = x$, and it is 0 otherwise. Therefore,

$$\mathrm{Tr}(\rho(g)) = \#\{x \in X : gx = x\} = \#X^g.$$

Next, we describe the averaging operator in the group algebra. Let

$$a := \sum_{g \in G} a_g g \in \mathbb{C}[G].$$

Define

$$\rho(a) = \sum_{a \in G} a_g \rho(g).$$

More generally, if π is any representation of G, define

$$\pi(a) = \sum_{g \in G} a_g \pi(g).$$

Theorem (Finite group trace formula). Let G act on a finite set X, and let V be the vector space of complex-valued functions on X. By Maschke's theorem,

$$V \cong \bigoplus_{\pi \in \widehat{G}} m_\pi \pi,$$

where $\widehat{G}$ is the set of irreducible complex representations of G, and $m_\pi = \dim_{\mathbb{C}} \operatorname{Hom}_G(\pi, V)$. Then, for every

$$a \in \sum_{g \in G} a_g g \in \mathbb{C}[G],$$

we have

$$\sum_{\pi \in \widehat{G}} m_\pi \operatorname{Tr}(\pi(a)) = \sum_{g \in G} a_g \# X^g.$$

Proof First, compute $\operatorname{Tr}(\rho(a))$ spectrally. Since V decomposes into irreducible representations, the operator $\rho(a)$ decomposes as

$$\rho(a) \cong \bigoplus_{\pi \in \widehat{G}} m_\pi \pi(a).$$

Hence,

$$\operatorname{Tr}(\rho(a)) = \sum_{\pi \in \widehat{G}} m_\pi \operatorname{Tr}(\pi(a)).$$

Now, compute the same trace another way. By linearity,

$$\operatorname{Tr}(\rho(a)) = \operatorname{Tr}\left(\sum_{g \in G} a_g \rho(g)\right) = \sum_{g \in G} a_g \operatorname{Tr}(\rho(g)).$$

From our prior discussion,

$$\operatorname{Tr}(\rho(g)) = \# X^g.$$

Therefore,

$$\operatorname{Tr}(\rho(a)) = \sum_{g \in G} a_g \# X^g.$$

Equating the two expressions for $\operatorname{Tr}(\rho(a))$ yields the desired result. $\square$

As a corollary, we obtain Burnside's lemma:

$$\#(G \setminus X) = \frac{1}{\# G} \sum_{g \in G} \# X^g.$$

Proof Let

$$e_G := \frac{1}{\#G} \sum_{g \in G} g \in \mathbb{C}[G].$$

Then, $\rho(e_G)$ is the projection $V \to V^G$ onto the G-invariant functions. Indeed, for $f \in V$,

$$\rho(e_G) f = \frac{1}{\#G} \sum_{g \in G} \rho(g) f.$$

This average is G-invariant. If $f \in V^G$, then $\rho(g) f = f$ for all g, so $\rho(e_G) f = f$.

Therefore,

$$\mathrm{Tr}(\rho(e_G)) = \dim V^G.$$

But, V^G is the space of functions on the orbit set $G \setminus X$, so

$$\dim V^G = \#(G \setminus X).$$

On the other hand, by the trace formula,

$$\mathrm{Tr}(\rho(e_G)) = \frac{1}{\#G} \sum_{g \in G} \# X^g.$$

Hence,

$$\#(G \setminus X) = \frac{1}{\#G} \sum_{g \in G} \# X^g.$$

□

Lefschetz trace formula over finite fields

We may turn the previous theorem arithmetic when the finite set is the set of geometric points of a variety, and when the map is the Frobenius.

Let X be a scheme of finite type over a finite field $\mathbb{F}_q$. Write $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. The geometric Frobenius $\mathrm{Frob}_q : \overline{X} \to \overline{X}$ acts on ℓ-adic cohomology $H^i_c(\overline{X}, \mathbb{Q}_\ell)$, where $\ell \neq \mathrm{char}(\mathbb{F}_q)$. First, we discuss the finite étale case. Suppose $X/\mathbb{F}_q$ is finite étale. Then, we claim that

$$\# X(\mathbb{F}_q) = \mathrm{Tr}(\mathrm{Frob}_q \mid H^0(\overline{X}, \mathbb{Q}_\ell)).$$

Proof Because of the finite étale condition, $\overline{X}$ is a finite discrete set of points. Write $S = \overline{X}(\overline{\mathbb{F}}_q)$. Then,

$$H^0(\overline{X}, \mathbb{Q}_\ell) \cong \mathrm{Fun}(S, \mathbb{Q}_\ell).$$

Moreover, $H^i(\overline{X}, \mathbb{Q}_\ell) = 0$ for $i > 0$, because $\overline{X}$ is a finite discrete space in the étale topology.

The rational points of X are exactly the geometric points fixed by the Frobenius. By the trace formula for finite sets,

$$\operatorname{Tr}(\mathrm{Frob}_q \mid \mathrm{Fun}(S, \mathbb{Q}_\ell)) = \# S^{\mathrm{Frob}_q} = \# X(\mathbb{F}_q),$$

as desired. ☐

Theorem (Grothendieck-Lefschetz trace formula). Let $X/\mathbb{F}_q$ be separated of finite type. Then,

$$\# X(\mathbb{F}_q) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\mathrm{Frob}_q \mid H_c^i(\overline{X}, \mathbb{Q}_\ell)).$$

The full proof requires the étale Lefschetz-Verdier fixed-point formula, which is roughly the assertion that for a correspondence $u : Y \to Y$ satisfying appropriate transversality and properness hypotheses,

$$\sum_i (-1)^i \operatorname{Tr}(u^* \mid H_c^i(Y, \mathbb{Q}_\ell)) = \sum_{x \in \mathrm{Fix}(u)} \mathrm{loc}_x(u),$$

where $\mathrm{loc}_x(u)$ is a local intersection-theoretic contribution at the fixed point x.

Now, apply this to $Y = \overline{X}$ and $u = \mathrm{Frob}_q$. As noted before, the fixed points of geometric Frobenius are exactly the rational points over $\mathbb{F}_q$. For Frobenius on a separated scheme of finite type over a finite field, the graph of the Frobenius intersects the diagonal with local multiplicity 1 at every fixed point; the result follows. Here, the idea is that fixed points relate to the alternating trace on cohomology.

Now, we provide two examples. The first is the affine line.

Let $X = \mathbb{A}^1_{\mathbb{F}_q}$. Then

$$X(\mathbb{F}_q) = \mathbb{F}_q,$$

so

$$\# X(\mathbb{F}_q) = q.$$

We compute the right-hand side of the Grothendieck-Lefschetz trace formula. Since X is not proper, we use compactly supported étale cohomology. Over $\overline{\mathbb{F}}_q$, we claim that

$$H_c^0(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0,$$

$$H_c^1(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0,$$

and

$$H_c^2(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

Proof Consider the open immersion

$$j : \mathbb{A}^1_{\overline{\mathbb{F}}_q} \hookrightarrow \mathbb{P}^1_{\overline{\mathbb{F}}_q}$$

with closed complement

$$i : \{\infty\} \hookrightarrow \mathbb{P}^1_{\mathbb{F}_q}.$$

The long exact sequence for compactly supported cohomology gives

$$0 \to H^0_c(\mathbb{A}^1, \mathbb{Q}_\ell) \to H^0(\mathbb{P}^1, \mathbb{Q}_\ell) \to H^0(\{\infty\}, \mathbb{Q}_\ell) \to H^1_c(\mathbb{A}^1, \mathbb{Q}_\ell) \to H^1(\mathbb{P}^1, \mathbb{Q}_\ell)$$

Now

$$H^0(\mathbb{P}^1, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell,$$

and

$$H^0(\{\infty\}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$

The restriction map

$$H^0(\mathbb{P}^1, \mathbb{Q}_\ell) \to H^0(\{\infty\}, \mathbb{Q}_\ell)$$

is an isomorphism. Hence

$$H^0_c(\mathbb{A}^1, \mathbb{Q}_\ell) = 0$$

and

$$H^1_c(\mathbb{A}^1, \mathbb{Q}_\ell) = 0,$$

because

$$H^1(\mathbb{P}^1, \mathbb{Q}_\ell) = 0.$$

Continuing the long exact sequence, we have

$$0 \to H^2_c(\mathbb{A}^1, \mathbb{Q}_\ell) \to H^2(\mathbb{P}^1, \mathbb{Q}_\ell) \to H^2(\{\infty\}, \mathbb{Q}_\ell).$$

Since $\{\infty\}$ is a point,

$$H^2(\{\infty\}, \mathbb{Q}_\ell) = 0.$$

Therefore

$$H^2_c(\mathbb{A}^1, \mathbb{Q}_\ell) \cong H^2(\mathbb{P}^1, \mathbb{Q}_\ell).$$

Finally,

$$H^2(\mathbb{P}^1_{\mathbb{F}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1),$$

so

$$H^2_c(\mathbb{A}^1_{\mathbb{F}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

$\square$

The geometric Frobenius acts on $\mathbb{Q}_\ell(-1)$ by multiplication by q. Hence

$$\mathrm{Tr}(\mathrm{Frob}_q \mid H^2_c(\mathbb{A}^1_{\mathbb{F}_q}, \mathbb{Q}_\ell)) = q.$$

Therefore,

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob}_q \mid H^i_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = q.$$

Thus the Grothendieck-Lefschetz trace formula gives

$$\#\mathbb{A}^1(\mathbb{F}_q) = q.$$

Next, we turn to the projective line for our second example.

Let $X = \mathbb{P}^1_{\mathbb{F}_q}$. Then

$$\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\},$$

so

$$\#\mathbb{P}^1(\mathbb{F}_q) = q + 1.$$

Since $\mathbb{P}^1$ is proper, compactly supported cohomology agrees with ordinary étale cohomology:

$$H^i_c(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = H^i(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell).$$

The cohomology groups are

$$H^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell,$$

$$H^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0,$$

and

$$H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

Proof The group $H^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ consists of locally constant $\mathbb{Q}_\ell$-valued functions on the connected scheme $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$, so

$$H^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$

The étale cohomology of projective space is

$$H^i(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0$$

for odd i, and

$$H^{2r}(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-r)$$

for $r = 0, 1$. Hence

$$H^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0$$

and

$$H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

□

The geometric Frobenius acts on H^0 by the identity, so

$$\operatorname{Tr}(\operatorname{Frob}_q \mid H^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = 1.$$

It acts on

$$H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1)$$

by multiplication by q, so

$$\operatorname{Tr}(\operatorname{Frob}_q \mid H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = q.$$

Therefore,

$$\sum_i (-1)^i \operatorname{Tr}(\operatorname{Frob}_q \mid H^i(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = 1 - 0 + q = q + 1.$$

Thus, the Grothendieck-Lefschetz trace formula gives

$$\#\mathbb{P}^1(\mathbb{F}_q) = q + 1.$$

Compact automorphic trace formula

In this section, we get a first glimpse of the utility of the trace formula in studying automorphic forms. We deal with the compact setting, as the noncompact case requires Arthur's truncation and continuous spectrum.

Let G be a connected real reductive Lie group, $\Gamma \subseteq G$ a discrete cocompact subgroup (sometimes referred to as a lattice), $M = \Gamma \setminus G$, dg a fixed right Haar measure on G with induced quotient measure $d\dot{g}$ on $\Gamma \setminus G$, and $f \in C_c^\infty(G)$.

Define the right regular representation

$$R : G \to U(L^2(\Gamma \setminus G))$$

by

$$(R(h)\varphi)(\Gamma g) = \varphi(\Gamma g h).$$

Then, define

$$R(f) = \int_G f(h) R(h) dh;$$

explicitly,

$$(R(f)\varphi)(\Gamma g) = \int_G f(h)\varphi(\Gamma g h) dh.$$

For $x \in \Gamma g_x$ and $y \in \Gamma g_y$, define the automorphic kernel

$$K_f(x, y) = \sum_{\gamma \in \Gamma} f(g_x^{-1} \gamma g_y).$$

It is not difficult to check that this expression is independent of the choice of representatives g_x, g_y.

Proof Suppose that $g_x' = \alpha g_x$ and $g_y' = \beta g_y$ for some $\alpha, \beta \in \Gamma$. Then

$$(g_x')^{-1}\gamma g_y' = g_x^{-1}\alpha^{-1}\gamma\beta g_y.$$

As γ ranges over Γ, the element $\alpha^{-1}\gamma\beta$ also ranges over Γ. Therefore,

$$\sum_{\gamma\in\Gamma} f((g_x')^{-1}\gamma g_y') = \sum_{\gamma\in\Gamma} f(g_x^{-1}\gamma g_y),$$

so K_f is well-defined on $(\Gamma \setminus G) \times (\Gamma \setminus G)$. □

We also need to know that the sum defining K_f is locally finite. Let $C = \operatorname{supp}(f)$, which is compact. The summand indexed by γ is nonzero only if

$$g_x^{-1}\gamma g_y \in C,$$

or equivalently,

$$\gamma \in g_x C g_y^{-1}.$$

Since $g_x C g_y^{-1}$ is compact and Γ is discrete, the intersection

$$\Gamma \cap g_x C g_y^{-1}$$

is finite. Hence, for fixed g_x, g_y, only finitely many terms contribute.

Moreover, if g_x and g_y vary in compact subsets $A, B \subset G$, then any contributing γ lies in

$$ACB^{-1},$$

which is compact. Hence

$$\Gamma \cap ACB^{-1}$$

is finite. Since $\Gamma \setminus G$ is compact, one may choose a compact set of representatives for all points of $\Gamma \setminus G$. It follows that the sum defining K_f is locally uniformly finite. Since each summand is smooth, K_f is a smooth function on $(\Gamma \setminus G) \times (\Gamma \setminus G)$.

Lemma (Kernel formula). For every smooth function φ on $\Gamma \setminus G$,

$$(R(f)\varphi)(x) = \int_{\Gamma\setminus G} K_f(x,y)\varphi(y)dy.$$

Proof Let $x = \Gamma g_x$. Then

$$\int_{\Gamma\setminus G} K_f(x,y)\varphi(y)dy = \int_{\Gamma\setminus G} \sum_{\gamma\in\Gamma} f(g_x^{-1}\gamma g_y)\varphi(\Gamma g_y)d(\Gamma g_y).$$

By local finiteness, we may interchange the sum and the integral. We use the quotient integration formula

$$\int_{\Gamma\setminus G} \sum_{\gamma\in\Gamma} F(\gamma g)d(\Gamma g) = \int_G F(u)du$$

for compactly supported continuous functions F on G. Apply this to

$$F(u) = f(g_x^{-1}u)\varphi(\Gamma u).$$

Then

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g_x^{-1}\gamma g_y)\varphi(\Gamma g_y)d(\Gamma g_y) = \int_G f(g_x^{-1}u)\varphi(\Gamma u)du.$$

Now set $u = g_x h$. Since G is unimodular, $du = dh$, and therefore

$$\int_G f(g_x^{-1}u)\varphi(\Gamma u)du = \int_G f(h)\varphi(\Gamma g_x h)dh.$$

By definition, the right-hand side is

$$(R(f)\varphi)(\Gamma g_x).$$

Thus,

$$(R(f)\varphi)(x) = \int_{\Gamma\backslash G} K_f(x,y)\varphi(y)dy.$$

□

The diagonal trace

We now compute the trace of $R(f)$ by restricting its kernel to the diagonal. Since $\Gamma \setminus G$ is compact and K_f is smooth, $R(f)$ is a smoothing operator on a compact manifold. Hence it is trace class.

Lemma (Trace of a smoothing operator). Let M be a compact smooth manifold with smooth measure dx. Let A be a smoothing operator on $L^2(M)$ with smooth kernel $K_A(x,y)$. Then A is trace class and

$$\mathrm{Tr}(A) = \int_M K_A(x,x)dx.$$

Proof Choose a positive self-adjoint elliptic differential operator Δ on M, and let $\{e_j\}_{j\geq 0}$ be an orthonormal basis of $L^2(M)$ consisting of smooth eigenfunctions:

$$\Delta e_j = \lambda_j e_j.$$

Since A is smoothing, for every $N \geq 0$, the operator

$$(1+\Delta)^N A$$

is bounded on $L^2(M)$. Hence the matrix coefficients

$$\langle Ae_j, e_k\rangle$$

decay rapidly in j and k. In particular,

$$\sum_j |\langle Ae_j, e_j\rangle| < \infty,$$

so A is trace class and

$$\mathrm{Tr}(A) = \sum_j \langle Ae_j, e_j \rangle.$$

Using the kernel,

$$\langle Ae_j, e_j \rangle = \int_M \int_M K_A(x,y) e_j(y) \overline{e_j(x)} \, dy \, dx.$$

Thus,

$$\mathrm{Tr}(A) = \sum_j \int_M \int_M K_A(x,y) e_j(y) \overline{e_j(x)} \, dy \, dx.$$

By rapid convergence, we may interchange the sum and the integrals. Since

$$\sum_j e_j(y) \overline{e_j(x)} = \delta_x(y)$$

as distributions, we obtain

$$\mathrm{Tr}(A) = \int_M \int_M K_A(x,y) \delta_x(y) \, dy \, dx = \int_M K_A(x,x) dx.$$

□

Applying this to the automorphic kernel gives

$$\mathrm{Tr}(R(f)) = \int_{\Gamma \backslash G} K_f(x,x) dx.$$

If $x = \Gamma g$, then

$$K_f(x,x) = \sum_{\gamma \in \Gamma} f(g^{-1} \gamma g).$$

Therefore,

$$\mathrm{Tr}(R(f)) = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(g^{-1} \gamma g) d(\Gamma g).$$

Unfolding and conjugacy classes

We now rewrite the last expression as a sum over conjugacy classes in Γ.

For $\gamma \in \Gamma$, define

$$\Gamma_\gamma = \{\delta \in \Gamma : \delta\gamma = \gamma\delta\},$$

and

$$G_\gamma = \{h \in G : h\gamma = \gamma h\}.$$

Lemma (Conjugacy classes and centralizers). For fixed $\gamma \in \Gamma$, the map

$$\Gamma_\gamma \setminus \Gamma \to \{\Gamma\text{-conjugates of } \gamma\}$$

defined by

$$\Gamma_\gamma\delta \mapsto \delta^{-1}\gamma\delta$$

is a bijection.

Proof First, the map is well-defined. If $\delta_1 = \eta\delta_2$ with $\eta \in \Gamma_\gamma$, then

$$\delta_1^{-1}\gamma\delta_1 = \delta_2^{-1}\eta^{-1}\gamma\eta\delta_2 = \delta_2^{-1}\gamma\delta_2.$$

The map is clearly surjective. For injectivity, suppose

$$\delta_1^{-1}\gamma\delta_1 = \delta_2^{-1}\gamma\delta_2.$$

Multiplying on the left by δ_2 and on the right by δ_1^{-1} gives

$$\delta_2\delta_1^{-1}\gamma = \gamma\delta_2\delta_1^{-1}.$$

Hence $\delta_2\delta_1^{-1} \in \Gamma_\gamma$, which is equivalent to

$$\Gamma_\gamma\delta_1 = \Gamma_\gamma\delta_2.$$

Thus the map is injective. □

Lemma (Unfolding). Let $\Gamma' \subset \Gamma$ be a subgroup. For a locally finite sum, one has

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma'\backslash\Gamma} \Phi(\delta g)d(\Gamma g) = \int_{\Gamma'\backslash G} \Phi(g)d(\Gamma' g).$$

Proof The natural map

$$\Gamma' \setminus G \to \Gamma \setminus G$$

has fiber over Γg naturally identified with $\Gamma' \setminus \Gamma$. Indeed, the points above Γg are represented by

$$\Gamma'\delta g,$$

where $\delta \in \Gamma$, and

$$\Gamma'\delta_1 g = \Gamma'\delta_2 g$$

if and only if $\delta_1\delta_2^{-1} \in \Gamma'$. Therefore, integration over $\Gamma' \setminus G$ may be computed by integrating over $\Gamma \setminus G$ and summing over the fibers:

$$\int_{\Gamma'\backslash G} \Phi(g)d(\Gamma' g) = \int_{\Gamma\backslash G} \sum_{\delta\in\Gamma'\backslash\Gamma} \Phi(\delta g)d(\Gamma g).$$

□

Theorem (Geometric expansion). For $f \in C_c^\infty(G)$,

$$\mathrm{Tr}(R(f)) = \sum_{[\gamma]_\Gamma} \int_{\Gamma_\gamma\backslash G} f(g^{-1}\gamma g)d(\Gamma_\gamma g),$$

where $[\gamma]_\Gamma$ runs over Γ-conjugacy classes.

Proof We have

$$\mathrm{Tr}(R(f)) = \int_{\Gamma\backslash G} \sum_{\eta\in\Gamma} f(g^{-1}\eta g)d(\Gamma g).$$

Decompose Γ into conjugacy classes:

$$\Gamma = \bigsqcup_{[\gamma]_\Gamma} \{\delta^{-1}\gamma\delta : \delta \in \Gamma_\gamma \setminus \Gamma\}.$$

Therefore,

$$\sum_{\eta\in\Gamma} f(g^{-1}\eta g) = \sum_{[\gamma]_\Gamma} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g).$$

Hence,

$$\mathrm{Tr}(R(f)) = \sum_{[\gamma]_\Gamma} \int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g)d(\Gamma g).$$

For fixed γ, define

$$\Phi_\gamma(g) = f(g^{-1}\gamma g).$$

Then

$$\Phi_\gamma(\delta g) = f((\delta g)^{-1}\gamma(\delta g)) = f(g^{-1}\delta^{-1}\gamma\delta g).$$

By the unfolding lemma,

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} \Phi_\gamma(\delta g)d(\Gamma g) = \int_{\Gamma_\gamma\backslash G} \Phi_\gamma(g)d(\Gamma_\gamma g).$$

Thus,

$$\mathrm{Tr}(R(f)) = \sum_{[\gamma]_\Gamma} \int_{\Gamma_\gamma\backslash G} f(g^{-1}\gamma g)d(\Gamma_\gamma g).$$

□

For $\gamma \in G$, define the orbital integral of f at γ by

$$O_\gamma(f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)d(G_\gamma g),$$

whenever the integral converges.

Lemma (Orbital-integral form). Assume that $\Gamma_\gamma \setminus G_\gamma$ has finite volume. Then

$$\int_{\Gamma_\gamma\backslash G} f(g^{-1}\gamma g)d(\Gamma_\gamma g) = \mathrm{vol}(\Gamma_\gamma \setminus G_\gamma)O_\gamma(f).$$

Proof The natural map

$$\Gamma_\gamma \setminus G \to G_\gamma \setminus G$$

has fiber $\Gamma_\gamma \setminus G_\gamma$. The function

$$g \mapsto f(g^{-1}\gamma g)$$

is left G_γ-invariant, since for $h \in G_\gamma$,

$$(hg)^{-1}\gamma(hg) = g^{-1}h^{-1}\gamma hg = g^{-1}\gamma g.$$

Thus the integral over $\Gamma_\gamma \setminus G$ factors as the volume of the fiber times the integral over $G_\gamma \setminus G$:

$$\int_{\Gamma_\gamma \setminus G} f(g^{-1}\gamma g)d(\Gamma_\gamma g) = \mathrm{vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} f(g^{-1}\gamma g)d(G_\gamma g).$$

The last integral is $O_\gamma(f)$. □

The spectral side

Since $\Gamma \setminus G$ is compact, the right regular representation decomposes discretely:

$$L^2(\Gamma \setminus G) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} m_\pi \mathcal{H}_\pi,$$

where $\widehat{G}$ denotes the unitary dual of G, $\mathcal{H}_\pi$ is the Hilbert space of π, and

$$m_\pi = \dim \mathrm{Hom}_G(\mathcal{H}_\pi, L^2(\Gamma \setminus G)).$$

The multiplicities m_π are finite.

For an irreducible unitary representation π of G, define

$$\pi(f) = \int_G f(g)\pi(g)dg.$$

Theorem (Spectral expansion). For $f \in C_c^\infty(G)$,

$$\mathrm{Tr}(R(f)) = \sum_{\pi \in \widehat{G}} m_\pi \,\mathrm{Tr}(\pi(f)).$$

Proof Under the discrete decomposition

$$L^2(\Gamma \setminus G) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} m_\pi \mathcal{H}_\pi,$$

the operator $R(f)$ decomposes as

$$R(f) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} m_\pi \pi(f).$$

Since $R(f)$ is trace class,

$$\mathrm{Tr}(R(f)) = \sum_{\pi \in \widehat{G}} m_\pi \,\mathrm{Tr}(\pi(f)).$$

□

Theorem (Compact automorphic trace formula). Let G be a connected real reductive Lie group, let $\Gamma \subset G$ be a cocompact discrete subgroup, and let $f \in C_c^\infty(G)$. Then

$$\sum_{\pi \in \widehat{G}} m_\pi \,\mathrm{Tr}(\pi(f)) = \sum_{[\gamma]_\Gamma} \int_{\Gamma_\gamma \setminus G} f(g^{-1}\gamma g)d(\Gamma_\gamma g).$$

Equivalently, when the centralizer quotients have finite volume,

$$\sum_{\pi \in \widehat{G}} m_\pi \operatorname{Tr}(\pi(f)) = \sum_{[\gamma]_\Gamma} \operatorname{vol}(\Gamma_\gamma \setminus G_\gamma) O_\gamma(f).$$

Proof Both sides are equal to $\operatorname{Tr}(R(f))$. The first equality follows by comparing the spectral expansion with the geometric expansion. The orbital-integral form follows from the preceding lemma. □

Compact hyperbolic surfaces

Let

$$G = \mathrm{PSL}_2(\mathbb{R}), \qquad K = \mathrm{SO}(2),$$

and let $\Gamma \subset G$ be torsion-free and cocompact. Then

$$Y = \Gamma \setminus \mathbb{H} \cong \Gamma \setminus G/K$$

is a compact hyperbolic surface.

Let

$$f \in C_c^\infty(K \setminus G/K)$$

be K-bi-invariant. Then $R(f)$ preserves

$$L^2(\Gamma \setminus G)^K \cong L^2(\Gamma \setminus G/K) \cong L^2(Y).$$

Let Δ be the positive hyperbolic Laplacian on Y. Since Y is compact, its spectrum is discrete:

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots,$$

with $\lambda_j \to \infty$. Write

$$\lambda_j = \frac{1}{4} + r_j^2.$$

For $f \in C_c^\infty(K \setminus G/K)$, define its spherical transform by

$$h_f(r) = \int_G f(g)\varphi_r(g)dg,$$

where φ_r is the spherical function normalized by $\varphi_r(1) = 1$.

Theorem (Spectral side for a compact hyperbolic surface). For $f \in C_c^\infty(K \setminus G/K)$,

$$\operatorname{Tr}(R(f)|L^2(Y)) = \sum_{j=0}^{\infty} h_f(r_j).$$

Proof Since Y is compact, there is an orthonormal basis $\{\phi_j\}_{j \ge 0}$ of $L^2(Y)$ consisting of smooth Laplace eigenfunctions:

$$\Delta \phi_j = \lambda_j \phi_j.$$

The operator $R(f)$ commutes with the action of the algebra of G-invariant differential operators on G/K, hence with Δ. Thus it preserves each Laplace eigenspace.

On the spherical representation generated by an eigenfunction with spectral parameter r_j, the K-fixed line is acted on by the scalar $h_f(r_j)$. Therefore

$$R(f)\phi_j = h_f(r_j)\phi_j.$$

Since $R(f)$ is trace class,

$$\mathrm{Tr}(R(f)|L^2(Y)) = \sum_{j=0}^{\infty}\langle R(f)\phi_j, \phi_j\rangle = \sum_{j=0}^{\infty} h_f(r_j).$$

□

Every nontrivial element of Γ is hyperbolic. Indeed, a torsion-free subgroup contains no nontrivial elliptic elements, and a cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$ contains no parabolic elements.

For a hyperbolic element $\gamma \in \Gamma$, let $\ell(\gamma)$ denote its translation length on $\mathbb{H}$.

Lemma (Centralizers in a cocompact Fuchsian group). Let $\gamma \in \Gamma$ be nontrivial. Then there is a unique primitive element $\gamma_0 \in \Gamma$ and an integer $n \geq 1$ such that

$$\gamma = \gamma_0^n,$$

and

$$\Gamma_\gamma = \langle \gamma_0 \rangle.$$

Moreover,

$$\ell(\gamma) = n\ell(\gamma_0).$$

Proof The element γ is hyperbolic, hence has two fixed points on $\partial\mathbb{H}$. Its axis $\mathrm{Ax}(\gamma)$ is the geodesic joining these two points. If $\delta \in \Gamma_\gamma$, then $\delta\gamma = \gamma\delta$, so δ preserves the fixed-point set of γ, and hence preserves $\mathrm{Ax}(\gamma)$.

Thus Γ_γ acts by translations on $\mathrm{Ax}(\gamma) \cong \mathbb{R}$. Since Γ is discrete, the image of Γ_γ in the translation group of $\mathbb{R}$ is a discrete subgroup of $\mathbb{R}$. Since it contains γ, it is nontrivial, hence infinite cyclic. Let γ_0 be the generator translating in the same direction as γ with minimal positive translation length. Then

$$\Gamma_\gamma = \langle \gamma_0 \rangle,$$

and $\gamma = \gamma_0^n$ for some $n \geq 1$. Translation length is additive under powers along the same axis, so

$$\ell(\gamma) = n\ell(\gamma_0).$$

□

The identity conjugacy class contributes

$$\int_{\Gamma\backslash G} f(g^{-1}1g)d(\Gamma g) = \mathrm{vol}(\Gamma \setminus G)f(1).$$

For the standard Haar normalization,

$$f(1) = \frac{1}{4\pi}\int_{-\infty}^{\infty} h_f(r)r\tanh(\pi r)dr.$$

Thus the identity contribution is

$$\frac{\mathrm{Area}(Y)}{4\pi}\int_{-\infty}^{\infty} h_f(r)r\tanh(\pi r)dr.$$

For $\gamma \neq 1$, the quotient $\Gamma_\gamma \setminus G_\gamma$ has volume $\ell(\gamma_0)$. The orbital integral computation for $G = \mathrm{PSL}_2(\mathbb{R})$ gives

$$O_\gamma(f) = \frac{g_f(\ell(\gamma))}{2\sinh(\ell(\gamma)/2)},$$

where

$$g_f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_f(r)e^{-iru}dr.$$

Therefore the contribution of $[\gamma] \neq [1]$ is

$$\frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)}g_f(\ell(\gamma)).$$

Theorem (Selberg trace formula, compact torsion-free case). Let $Y = \Gamma \setminus \mathbb{H}$ be a compact hyperbolic surface. Let h be an even Paley-Wiener function, and let

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}dr.$$

Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

be the eigenvalues of the positive Laplacian on Y, written as

$$\lambda_j = \frac{1}{4} + r_j^2.$$

Then

$$\sum_{j=0}^{\infty} h(r_j) = \frac{\mathrm{Area}(Y)}{4\pi}\int_{-\infty}^{\infty} h(r)r\tanh(\pi r)dr + \sum_{[\gamma]\neq[1]} \frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)}g(\ell(\gamma)).$$

Here $[\gamma]$ runs over nontrivial conjugacy classes in Γ, and γ_0 is the primitive element underlying γ.

Proof Choose $f \in C_c^\infty(K \setminus G/K)$ whose spherical transform is h. By the spectral-side computation,

$$\operatorname{Tr}(R(f)|L^2(Y)) = \sum_{j=0}^{\infty} h(r_j).$$

By the compact automorphic trace formula,

$$\operatorname{Tr}(R(f)) = \sum_{[\gamma]_\Gamma} \int_{\Gamma_\gamma \backslash G} f(g^{-1}\gamma g)\, d(\Gamma_\gamma g).$$

The identity conjugacy class contributes

$$\frac{\operatorname{Area}(Y)}{4\pi} \int_{-\infty}^{\infty} h(r) r \tanh(\pi r) dr.$$

Each nontrivial conjugacy class contributes

$$\frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)} g(\ell(\gamma)).$$

Substituting these contributions into the trace formula gives

$$\sum_{j=0}^{\infty} h(r_j) = \frac{\operatorname{Area}(Y)}{4\pi} \int_{-\infty}^{\infty} h(r) r \tanh(\pi r) dr + \sum_{[\gamma]\neq[1]} \frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)} g(\ell(\gamma)).$$

□

The adelic compact trace formula

Let F be a number field, let $\mathbb{A} = \mathbb{A}_F$, and let G be a connected reductive group over F. Suppose, for simplicity, that

$$G(F) \backslash G(\mathbb{A})$$

is compact. Let

$$f \in C_c^{\infty}(G(\mathbb{A})).$$

Define

$$R(f) = \int_{G(\mathbb{A})} f(h) R(h) dh$$

on $L^2(G(F) \backslash G(\mathbb{A}))$.

For $x = G(F)g_x$ and $y = G(F)g_y$, define

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(g_x^{-1}\gamma g_y).$$

The same proof as before shows that this is a well-defined kernel and that

$$\operatorname{Tr}(R(f)) = \int_{G(F)\backslash G(\mathbb{A})} K_f(x,x) dx.$$

Theorem (Compact adelic trace formula). Assume that $G(F) \backslash G(\mathbb{A})$ is compact. Then

$$\sum_{\pi} m_\pi \operatorname{Tr}(\pi(f)) = \sum_{[\gamma]} \operatorname{vol}(G_\gamma(F) \setminus G_\gamma(\mathbb{A})) O_\gamma(f),$$

where π runs over irreducible automorphic representations occurring in $L^2(G(F) \setminus G(\mathbb{A}))$, and

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} f(g^{-1}\gamma g) dg.$$

Proof The kernel computation gives

$$\operatorname{Tr}(R(f)) = \int_{G(F) \setminus G(\mathbb{A})} \sum_{\gamma \in G(F)} f(g^{-1}\gamma g) dg.$$

Decompose $G(F)$ into $G(F)$-conjugacy classes. For each γ, unfolding gives

$$\int_{G(F) \setminus G(\mathbb{A})} \sum_{\delta \in G_\gamma(F) \setminus G(F)} f(g^{-1}\delta^{-1}\gamma\delta g) dg = \int_{G_\gamma(F) \setminus G(\mathbb{A})} f(g^{-1}\gamma g) dg.$$

Since $g \mapsto f(g^{-1}\gamma g)$ is left $G_\gamma(\mathbb{A})$-invariant, this equals

$$\operatorname{vol}(G_\gamma(F) \setminus G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} f(g^{-1}\gamma g) dg.$$

Thus the geometric side is

$$\sum_{[\gamma]} \operatorname{vol}(G_\gamma(F) \setminus G_\gamma(\mathbb{A})) O_\gamma(f).$$

On the other hand, compactness gives the discrete spectral decomposition

$$L^2(G(F) \setminus G(\mathbb{A})) \cong \widehat{\bigoplus}_\pi m_\pi \pi.$$

Hence

$$\operatorname{Tr}(R(f)) = \sum_{\pi} m_\pi \operatorname{Tr}(\pi(f)).$$

Equating the two expressions gives the result. $\square$

Hecke operators as test functions

Let v be a finite place of F, and suppose G is unramified at v. Let

$$K_v = G(\mathcal{O}_v).$$

The spherical Hecke algebra is

$$\mathcal{H}(G(F_v), K_v) = C_c(K_v \setminus G(F_v)/K_v),$$

with convolution

$$(f_1 * f_2)(g) = \int_{G(F_v)} f_1(gh^{-1}) f_2(h) dh.$$

If

$$f_v = \mathbf{1}_{K_v a K_v},$$

then f_v defines a Hecke operator. For a decomposable global test function

$$f = \bigotimes_w f_w,$$

the operator $R(f)$ is a global Hecke operator when the local factors f_w are chosen in the appropriate local Hecke algebras.

Thus the trace formula computes traces of Hecke operators:

$$\operatorname{Tr}(R(f)) = \sum_{\pi} m_\pi \operatorname{Tr}(\pi(f)) = \sum_{[\gamma]} \operatorname{vol}(G_\gamma(F) \setminus G_\gamma(\mathbb{A})) O_\gamma(f).$$

The noncompact case

If $G(F) \setminus G(\mathbb{A})$ is not compact, the preceding argument fails in two places.

First, the diagonal integral

$$\int_{G(F)\setminus G(\mathbb{A})} K_f(x,x)dx$$

may diverge.

Second, the spectral decomposition of

$$L^2(G(F) \setminus G(\mathbb{A}))$$

contains continuous spectrum. The continuous spectrum is described by Eisenstein series attached to proper parabolic subgroups.

For example, for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the quotient

$$\Gamma \setminus \mathbb{H}$$

is noncompact. The Eisenstein series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \operatorname{Im}(\gamma z)^s$$

initially converges for $\operatorname{Re}(s) > 1$ and admits meromorphic continuation to $s \in \mathbb{C}$. These Eisenstein series account for the continuous spectrum of $L^2(\Gamma \setminus \mathbb{H})$.

Arthur’s trace formula modifies the compact trace formula by replacing the divergent kernel integral with a truncated kernel integral. The truncation removes divergent contributions from proper parabolic subgroups.

Arthur’s trace formula

Let F be a number field, let $\mathbb{A} = \mathbb{A}_F$, and let G be a connected reductive group over F. Let

$$f \in C_c^\infty(G(\mathbb{A})).$$

Arthur defines two distributions

$$J_{\mathrm{geom}}(f)$$

and

$$J_{\mathrm{spec}}(f)$$

such that

$$J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f).$$

In the compact case,

$$J_{\mathrm{geom}}(f) = \sum_{[\gamma]} \mathrm{vol}(G_\gamma(F) \setminus G_\gamma(\mathbb{A})) O_\gamma(f),$$

and

$$J_{\mathrm{spec}}(f) = \sum_{\pi} m_\pi \operatorname{Tr}(\pi(f)).$$

In the noncompact case, the geometric side is a sum of weighted orbital integrals:

$$J_{\mathrm{geom}}(f) = \sum_M \sum_\gamma a^M(\gamma) J_M(\gamma, f).$$

Here M ranges over Levi subgroups of G, the element γ ranges over suitable conjugacy classes in $M(F)$, the coefficient $a^M(\gamma)$ is global, and

$$J_M(\gamma, f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g) v_M(g) dg$$

is a weighted orbital integral. If $M = G$, then $v_G(g) = 1$, so this is an ordinary orbital integral.

The spectral side has the corresponding form

$$J_{\mathrm{spec}}(f) = \sum_M \int_{\Pi(M)} a^M(\pi) J_M(\pi, f) d\pi.$$

Here $\Pi(M)$ denotes a suitable space of automorphic representations of $M(\mathbb{A})$, and the distributions $J_M(\pi, f)$ are built from induced representations, Eisenstein series, and intertwining operators.

Theorem (Arthur trace formula, schematic form). Let G be a connected reductive group over a number field F. For every $f \in C_c^\infty(G(\mathbb{A}_F))$, Arthur defines distributions

$$J_{\text{geom}}(f)$$

and

$$J_{\text{spec}}(f)$$

such that

$$J_{\text{geom}}(f) = J_{\text{spec}}(f).$$

The geometric side is expressed in terms of weighted orbital integrals. The spectral side is expressed in terms of automorphic representations, Eisenstein series, and intertwining operators. In the compact case, the formula reduces to the compact adelic trace formula.

Proof sketch Start from the formal kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

on $G(F) \setminus G(\mathbb{A})$. The integral

$$\int_{G(F)\setminus G(\mathbb{A})} K_f(x,x)dx$$

need not converge. Arthur defines a truncation operator Λ^T, depending on a truncation parameter T, and studies

$$\int_{G(F)\setminus G(\mathbb{A})} (\Lambda^T K_f)(x,x)dx.$$

This integral converges.

On the geometric side, one decomposes the kernel by conjugacy classes and parabolic subgroups. The truncation produces weight functions, and hence weighted orbital integrals.

On the spectral side, one decomposes $L^2(G(F) \setminus G(\mathbb{A}))$ into cuspidal data and Eisenstein series induced from Levi subgroups. The truncation produces terms involving normalized intertwining operators.

Both expansions have asymptotic expressions in T. The equality of the constant terms gives

$$J_{\text{geom}}(f) = J_{\text{spec}}(f).$$

$\square$

Transfer and the fundamental lemma

Let H and G be connected reductive groups over F. A morphism of L-groups

$$^{L}H \to {}^{L}G$$

is expected to induce transfer of automorphic representations from H to G.

The trace formula gives an indirect method. Suppose one can choose test functions

$$f_H \in C_c^\infty(H(\mathbb{A})),$$

and

$$f_G \in C_c^\infty(G(\mathbb{A})),$$

such that

$$J_{\text{geom}}^H(f_H) = J_{\text{geom}}^G(f_G).$$

Then Arthur's trace formula gives

$$J_{\text{spec}}^H(f_H) = J_{\text{geom}}^H(f_H) = J_{\text{geom}}^G(f_G) = J_{\text{spec}}^G(f_G).$$

Thus equality of geometric distributions implies equality of spectral distributions.

At a nonarchimedean place v, the required geometric equality is reduced to identities of orbital integrals. A typical local identity has the form

$$SO_{\gamma_H}(f_v^H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G) O_{\gamma_G}(f_v).$$

Here SO_{γ_H} is a stable orbital integral, O_{γ_G} is an ordinary orbital integral, and $\Delta(\gamma_H, \gamma_G)$ is a transfer factor.

The fundamental lemma asserts that, for canonical local test functions, these orbital integrals match. Since global orbital integrals factor into local orbital integrals, the fundamental lemma furnishes the local input needed to compare global trace formulas.

Shimura varieties, cohomology, and trace formulas

We now explain why Shimura varieties provide a natural setting in which the Grothendieck-Lefschetz trace formula and the Arthur-Selberg trace formula meet. A Shimura variety is attached to a Shimura datum (G, X), where G is a reductive group over $\mathbb{Q}$ and X is a suitable $G(\mathbb{R})$-homogeneous space. If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup, the complex points of the associated Shimura variety are given by

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K)\,.$$

Thus a Shimura variety has two simultaneous descriptions. On the one hand, it is an algebraic variety, defined over a number field called the reflex field. On the other hand, its complex points are described by an adelic double quotient. This double nature is what makes Shimura varieties central in the comparison between geometry and automorphic forms.

The algebraic side gives Frobenius and étale cohomology. Let E be the reflex field of (G, X), and suppose that $\mathrm{Sh}_K(G, X)$ has good reduction at a prime $\mathfrak{p}$ of E. Let

$$\mathrm{Sh}_{K,\overline{\mathbb{F}}_{\mathfrak{p}}}$$

denote the geometric special fiber. The geometric Frobenius

$$\Phi_{\mathfrak{p}}$$

acts on the compactly supported étale cohomology groups

$$H^i_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_\ell).$$

Thus one can study the alternating trace

$$\mathrm{Tr}\left(\Phi^r_{\mathfrak{p}} \mid H^*_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_\ell)\right),$$

where

$$H^*_c = \sum_i (-1)^i H^i_c.$$

By the Grothendieck-Lefschetz trace formula, this alternating trace is expressed as a fixed-point count for $\Phi^r_{\mathfrak{p}}$ on the special fiber.

The adelic side gives Hecke correspondences. If

$$g \in G(\mathbb{A}_f),$$

then the double coset

$$KgK$$

defines a correspondence on $\mathrm{Sh}_K(G, X)$. More generally, a compactly supported locally constant function

$$f^p \in C^\infty_c(G(\mathbb{A}^p_f))$$

defines a linear combination of Hecke correspondences away from p. Combining this with Frobenius gives a Frobenius-Hecke correspondence

$$\Phi^r_{\mathfrak{p}} \times f^p$$

on the special fiber. Hence one is led to study traces of the form

$$\mathrm{Tr}\left(\Phi^r_{\mathfrak{p}} \times f^p \mid H^*_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_\ell)\right).$$

By Grothendieck-Lefschetz, this trace is a fixed-point count for the correspondence

$$\Phi_{\mathfrak{p}}^{r} \times f^{p}.$$

The representation-theoretic meaning of the same trace comes from the fact that the cohomology of Shimura varieties is governed by automorphic representations. Very schematically, automorphic representations π of $G(\mathbb{A})$ contribute to the cohomology through terms of the form

$$\pi_f^K \otimes H^*(\mathfrak{g}, K_\infty; \pi_\infty),$$

where π_f is the finite part of π, π_∞ is the archimedean part, and

$$H^*(\mathfrak{g}, K_\infty; \pi_\infty)$$

is relative Lie algebra cohomology. Thus the same Frobenius-Hecke trace should also admit a spectral expression in terms of automorphic representations.

This is the basic reason that Shimura varieties bring together the two trace formulas. The Grothendieck-Lefschetz trace formula gives

$$\operatorname{Tr}\left(\Phi_{\mathfrak{p}}^{r} \times f^{p} \mid H_c^*\right) = \#\operatorname{Fix}(\Phi_{\mathfrak{p}}^{r} \times f^{p}),$$

with local multiplicities understood. The automorphic description of the same cohomology suggests that the same trace should be expressible by the spectral side of an Arthur-Selberg trace formula for a carefully chosen adelic test function

$$f = f^p f_p f_\infty.$$

Here f^p records the Hecke correspondence away from p, the factor f_p records Frobenius and the geometry of the reduction at p, and the factor f_∞ is chosen to isolate the cohomological automorphic representations contributing to the Shimura variety.

The fixed-point side is already close to the geometric side of the Arthur-Selberg trace formula. The fixed points of the Frobenius-Hecke correspondence are described by group-theoretic data: rational conjugacy classes, local conjugacy classes away from p, and σ-conjugacy classes at p. In favorable cases, the Lefschetz fixed-point formula gives an expression of the form

$$\sum_{\text{Kottwitz data}} c(\gamma_0; \gamma, \delta)\, O_\gamma(f^p)\, TO_\delta(\phi_p).$$

Here $O_\gamma(f^p)$ is an ordinary orbital integral away from p, $TO_\delta(\phi_p)$ is a twisted orbital integral at p, and $c(\gamma_0; \gamma, \delta)$ is a global volume factor. Thus point-counting on the Shimura variety produces exactly the kind of orbital-integral expression that appears on the geometric side of the trace formula.

However, this comparison is not term-by-term at the level of the ordinary, unstabilized trace formula. The Lefschetz fixed-point formula naturally organizes the fixed-point data into stable or endoscopic packets. By contrast, the ordinary Arthur-Selberg trace formula is initially written in terms of ordinary conjugacy classes and ordinary orbital integrals. These two expressions are morally related, but they are not yet in the same language.

This is why stabilization is necessary. Stabilization rewrites the geometric side of the Arthur-Selberg trace formula in terms of stable orbital integrals and endoscopic contributions. Schematically, an ordinary geometric expression

$$\sum_{\gamma} a(\gamma) O_{\gamma}(f)$$

is replaced by a stable expression involving endoscopic groups H:

$$\sum_{H} \iota(G,H) SJ^{H}_{\text{geom}}(f^{H}).$$

Here f^H is a transfer of the test function f to the endoscopic group H, and SJ^{H}_{geom} denotes a stable geometric distribution.

The fundamental lemma is the local input that makes this stabilization possible. At a nonarchimedean place, it asserts that matching test functions have matching stable orbital integrals:

$$SO_{\gamma_H}(f^H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G) O_{\gamma_G}(f).$$

Here SO_{γ_H} is a stable orbital integral on an endoscopic group, O_{γ_G} is an ordinary orbital integral on G, and $\Delta(\gamma_H, \gamma_G)$ is a transfer factor. Since global trace formulae factor into local orbital integrals, these local identities are precisely what allow the global geometric sides to be compared.

Thus Shimura varieties close the circle between the two trace formulas. The Grothendieck-Lefschetz trace formula counts fixed points of Frobenius-Hecke correspondences on the special fibers of Shimura varieties. The Arthur-Selberg trace formula organizes automorphic representations through orbital integrals. Stabilization, made possible by the fundamental lemma, is the mechanism that lets these two trace formulas be compared. In this way, point counts on Shimura varieties become a bridge from Frobenius and étale cohomology to automorphic representations and the Langlands program.

Thank you for reading!