Cobordism and characteristic classes, part 1

akrishna168 • 24 May 2025

In this post, I'll discuss a result- of which I'll prove half- that will force me to go back and write at least one (hopefully many) post(s) on characteristic classes proper. The result is as follows:

Theorem. Let M be a closed, smooth n-manifold. Then, there exists a smooth, compact $(n+1)$-manifold B with boundary $\partial B = M$ if and only if all the Stiefel-Whitney classes of M are zero.

Proof ($\implies$). Let $[B] \in H_{n+1}(B, M; \mathbb{Z}/2)$ be the relative fundamental class of the pair (B, M). Then, writing $\partial : H_{n+1}(B, M; \mathbb{Z}/2) \to H_n(B, M; \mathbb{Z}/2)$ for the homological differential, we have $\partial[B] = [M]$.

Now, the tangent bundle TB is well-defined even up to the boundary, and $TB|_M \cong TM \oplus E$, where E is the trivial rank-one bundle giving the outward normal to ∂B. Thus, $w_i(TB)|_M = w_i(TM)$. It follows that any polynomial P in the Stiefel-Whitney classes of M is in the image of $i^* : H^n(B) \to H^n(M)$, so $P \in \ker(\delta : H^n(M) \to H^{n+1}(B, M))$. Thus,

$$\langle P, [M] \rangle = \langle P, \partial([B]) \rangle = \langle \delta(P), [B] \rangle = 0,$$

as was sought. $\square$

The converse is more difficult. As a corollary, note that M_1, M_2 are (unoriented) cobordant manifolds if and only if all of their Stiefel-Whitney classes concur.