Nonexistence of immersions

akrishna168 • 23 May 2025

To get back into the flow of posting, I'll talk about a curiosity in differential topology that is most organically proven using the machinery of characteristic classes.

An *immersion* between smooth manifolds is a smooth map $f : M \to N$ whose differential on tangent spaces $df_x : T_xM \to T_{f(x)}N$ is injective on each fiber. The Whitney Immersion Theorem states that any smooth n dimensional manifold admits an immersion into $\mathbb{R}^{2n-1}$ (and an embedding into $\mathbb{R}^{2n}$, which can be improved nontrivially to $\mathbb{R}^{2n-1}$). We may ask whether the dimension $2n - 1$ is optimal: that is, whether or not every smooth manifold M of dimension n embeds into $\mathbb{R}^m$ for some $m < 2n - 1$. This turns out not to be the case if n is not a power of 2, but using Stiefel-Whitney classes, we may show that the number $2n - 1$ is optimal if $n = 2^k$.

Theorem. For $n = 2^k$, real projective space $\mathbb{RP}^n$ does not immerse into $\mathbb{R}^{2n-2}$.

Proof. Suppose there existed an immersion $f : \mathbb{RP}^n \to \mathbb{R}^{2n-2}$. Then, df is a morphism of vector bundles which covers f and which factors through the pullback $f^*T\mathbb{R}^{2n-2}$. Because f is an immersion, its differential df is fiberwise injective, and the same is true for the induced map $T\mathbb{RP}^n \to f^*T\mathbb{R}^{2n-2}$. Since $T\mathbb{R}^{2n-2}$ is a trivial vector bundle, it follows that $f^*T\mathbb{R}^{2n-2} \cong E$, i.e. $T\mathbb{RP}^n$ is a subbundle of a trivial bundle E of rank $2n - 2$.

Now consider the normal bundle $\nu \to \mathbb{RP}^n$, defined as the orthogonal complement of $T\mathbb{RP}^n$ in $T\mathbb{R}^{2n-2}$. By this definition, we have $T\mathbb{RP}^n \oplus \nu = E$, implying that rk $\nu = n - 2$. Moreover, by the axioms for Stiefel-Whitney classes, we have $1 = w(E) = w(T\mathbb{RP}^n)w(\nu)$. However,

$$w(T\mathbb{RP}^n) = (1 + x)^n = (1 + x)(1 + x)^{2k} = (1 + x)(1^{2^k} + x^{2^k}) = 1 + x + x^n,$$

whose multiplicative inverse in the ring $\mathbb{Z}/2[x]/(x^{n+1})$ is $a = \sum_{i=0}^{n-1} x^i$. This contradicts the fact that rk $\nu = n - 2$. $\square$