

After some algebra...

written by heyredhat on Functor Network
original link: <https://functor.network/user/1704/entry/703>

Let $\Phi = \alpha I + \beta J$ for $\beta = (1 - \alpha) (\frac{1}{n})$. Let

$$\begin{aligned} A_{ijk} &= \eta(\delta_{ij} - \delta_{ik} - \delta_{jk}) \\ B[Y]_{ij} &= \sum_k (\delta_{ij}\delta_{ik} - A_{ijk})y_k \\ C[Y] &= P\Phi B[Y]P\Phi \\ \mathcal{L}[Y] &= \chi C[Y]. \end{aligned}$$

Then

$$\begin{aligned} B[Y]_{ij} &= \sum_k (\delta_{ij}\delta_{ik} - \eta[\delta_{ij} - \delta_{ik} - \delta_{jk}])y_k \\ &= \delta_{ij}y_i - \delta_{ij}\eta\bar{y} + \eta y_i + \eta y_j, \end{aligned}$$

and

$$\mathcal{L}[Y]_{ij} = \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\sigma_j)y_m - \alpha\eta\bar{y}P(E_i|\sigma_j) + \kappa y_i + \kappa y_j - \beta\kappa\bar{y} \right\},$$

where $\kappa = \alpha\eta + \beta + n\beta\eta = \beta + \eta$. Let $Q(E|\sigma) = \Phi P(E|\sigma)$ and $Q(E|\rho) = \Phi P(E|\rho)$. Moreover,

$$[\mathcal{L}[\rho]\Phi]_{ij} = \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i)Q(E_m|\sigma_j)P(E_m|\rho) - \alpha\eta Q(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa Q(E_j|\rho) - \beta\kappa \right\}.$$

Suppose we demand commutativity: $\mathcal{L}[\rho]\Phi P(E|\tau) = \mathcal{L}[\tau]\Phi P(E|\rho)$. We have

$$\begin{aligned} \mathcal{L}[\rho]\Phi P(E|\tau) &= \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\tau)P(E_m|\rho) - \alpha\eta P(E_i|\tau) + \kappa P(E_i|\rho) + \kappa\gamma^{-1}P(\rho|\tau) - \beta\kappa \right\} \\ \mathcal{L}[\tau]\Phi P(E|\rho) &= \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\rho)P(E_m|\tau) - \alpha\eta P(E_i|\rho) + \kappa P(E_i|\tau) + \kappa\gamma^{-1}P(\rho|\tau) - \beta\kappa \right\}, \end{aligned}$$

from which we conclude that $\eta = -\frac{\beta}{\alpha+1}$. Let us further assume self-duality for our reference states and effects so that invoking Bayes' rule, we have

$$P(\rho|\sigma_i) = P(\rho|E_i) = \frac{P(E_i|\rho)P(\rho)}{P(E_i)} = \gamma P(E_i|\rho), \quad (1)$$

so that $\gamma P(E|\rho) = P(\rho|\sigma)$. If we assume self-duality in general, then

$$P(\rho|\rho) = P(\rho|\sigma)\Phi P(E|\rho) = \gamma P(E|\rho)\Phi P(E|\rho)$$

$$= \gamma \left[\alpha \sum_i P(E_i|\rho)^2 + \beta \right].$$

If $P(\rho|\rho) = 1$, then

$$\sum_i P(E_i|\rho)^2 = \frac{1}{\alpha} [\gamma^{-1} - \beta].$$

In 3-design quantum theory, $\alpha = d + 1$, $\beta = -\frac{d}{n}$, and $\gamma = \frac{n}{d}$, which gives $\sum_i P(E_i|\rho)^2 = (\frac{d}{n}) \frac{2}{d+1}$.

Let us now consider $\tilde{P}(E|\rho^2) = \mathcal{L}[\rho]\Phi P(E|\rho)$, or

$$\tilde{P}(E_i|\rho^2) = \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + (\kappa - \alpha\eta) P(E_i|\rho) + \kappa\gamma^{-1} P(\rho|\rho) - \beta\kappa \right\}.$$

If we further demand that $\rho = \rho^2$, or $P(E_i|\rho) = \tilde{P}(E_i|\rho^2)$, then

$$P(E_i|\rho) = \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right) \left[\alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta) \right],$$

and since $\sum_i P(E_i|\rho) = 1$, we have

$$\left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right) \left[\alpha^2 \sum_m P(E_m|\rho)^2 + n\kappa(\gamma^{-1} - \beta) \right] = 1,$$

or $\alpha(\gamma^{-1} - \beta) + n\kappa(\gamma^{-1} - \beta) = \chi^{-1} - \kappa + \alpha\eta$, which fixes

$$\chi = \frac{\gamma(\alpha + 1)}{2\alpha}.$$

So by assuming **commutativity**, that is, $\mathcal{L}[\rho]\Phi P(E|\tau) = \mathcal{L}[\tau]\Phi P(E|\rho)$, we can fix η . By assuming **self-duality**, and for **pure states** that $P(\rho|\rho) = 1$ and $P(E|\rho) = \tilde{P}(E|\rho^2)$, we can fix $\sum_i P(E_i|\rho)^2$ and then χ . (Notice we haven't said anything about the extremal states as defined by the variance restriction alone.) Indeed, now

$$\mathcal{L}[Y]_{ij} = \frac{1}{2}\gamma \left\{ \alpha(\alpha + 1) \sum_m P(E_m|\sigma_i) P(E_m|\sigma_j) y_m - \left(\frac{\alpha - 1}{n} \right) (\bar{y}P(E_i|\sigma_j) + y_i + y_j) - \left(\frac{\alpha - 1}{n} \right)^2 \bar{y} \right\}.$$

We can go further using $P(\rho|\rho) = \gamma P(E|\rho)\Phi P(E|\rho) = 1$ by substituting in the expression for $\tilde{P}(E|\rho^2)$. Thus

$$\begin{aligned} 1 &= \gamma \sum_i Q(E_i|\rho) \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right) \left[\alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta) \right] \\ &= \left(\frac{\gamma}{\chi^{-1} - \kappa + \alpha\eta} \right) \left[\alpha^2 \sum_m P(E_m|\rho)^3 + \kappa(\gamma^{-1} - \beta) \right], \end{aligned}$$

so that

$$\begin{aligned}\sum_m P(E_m|\rho)^3 &= \frac{1}{\alpha^2} \left[\left(\frac{\chi^{-1} - \kappa + \alpha\eta}{\gamma} \right) - \kappa(\gamma^{-1} - \beta) \right] \\ &= \frac{((\alpha-1)\gamma + n)((\alpha-1)\gamma + 2n)}{\alpha(\alpha+1)\gamma^2 n^2}.\end{aligned}$$

In 3-design quantum theory, $\alpha = d+1$ and $\gamma = \frac{n}{d}$, and so $\sum_i P(E_i|\rho)^3 = \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}$. Can we keep going? No! For example, we also have

$$\begin{aligned}P(\rho|\rho) &= 1 \\ &= \sum_i \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right)^2 \left[\alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta) \right] \left[\alpha^2 \sum_q Q(E_q|\sigma_i) P(E_q|\rho)^2 + \kappa(\gamma^{-1} - \beta) \right] \\ &= \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right)^2 \left\{ \alpha^4 \sum_{mq} P(E_m|\rho)^2 P(E_q|\rho)^2 P(E_m|\sigma_q) + 2\alpha^2 \kappa(\gamma^{-1} - \beta) \sum_m P(E_m|\rho)^2 + n\kappa^2(\gamma^{-1} - \beta)^2 \right\} \\ &= \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta} \right)^2 \left\{ \alpha^4 \sum_{mq} P(E_m|\rho)^2 P(E_q|\rho)^2 P(E_m|\sigma_q) + 2\kappa\alpha(\gamma^{-1} - \beta)^2 + n\kappa^2(\gamma^{-1} - \beta)^2 \right\},\end{aligned}$$

so that

$$\begin{aligned}\sum_{mq} P(E_m|\rho)^2 P(E_m|\sigma_q) P(E_q|\rho)^2 &= \frac{1}{\alpha^4} \left\{ \gamma^{-1} (\chi^{-1} - \kappa + \alpha\eta)^2 - 2\kappa\alpha(\gamma^{-1} - \beta)^2 - n\kappa^2(\gamma^{-1} - \beta)^2 \right\} \\ &= \frac{((\alpha-1)\gamma + n)^2 ((\alpha-1)(\alpha+3)\gamma + 4n)}{\alpha^2(\alpha+1)^2\gamma^3 n^3},\end{aligned}$$

which for 3-design quantum theory becomes $\frac{4d^3(d+8)}{(d+1)^2(d+2)^2n^3}$, which is true! But the pure probability-assignments don't live on $(p > 3)$ -norm spheres. (Does the 2-norm and 3-norm sphere condition imply that $P(E|\rho) = \tilde{P}(E|\rho^2)$?)

It would be nice to fix γ in terms of α . The next thing, however, to consider is the Jordan identity itself, $\mathcal{L}[\rho]\Phi = \mathcal{L}[\rho^2]\Phi$. On the one hand,

$$[\mathcal{L}[\rho]\Phi]_{ij} = \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i) Q(E_m|\sigma_j) P(E_m|\rho) - \alpha\eta Q(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa Q(E_j|\rho) - \beta\kappa \right\}.$$

On the other hand,

$$\tilde{P}(E_i|\rho^2) = \chi \left\{ \alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + (\kappa - \alpha\eta) P(E_i|\rho) + \kappa\gamma^{-1} P(\rho|\rho) - \beta\kappa \right\}$$

and

$$\sum_i \tilde{P}(E_i|\rho^2) = \chi \left\{ \alpha^2 \sum_m P(E_m|\rho)^2 + n\kappa\gamma^{-1} P(\rho|\rho) + \alpha\beta \right\},$$

so that

$$\begin{aligned}
& \mathcal{L}[\rho^2]_{ij} \\
&= \chi \left\{ \alpha^4 \chi \sum_m P(E_m | \sigma_i) P(E_m | \sigma_j) \sum_p P(E_m | \sigma_p) P(E_p | \rho)^2 \right. \\
&\quad + \alpha^2 \chi (\kappa - \alpha \eta) \sum_m P(E_m | \sigma_i) P(E_m | \sigma_j) P(E_m | \rho) \\
&\quad + \alpha^2 \chi \kappa \sum_p P(E_p | \sigma_i) P(E_p | \rho)^2 \\
&\quad + \alpha^2 \chi \kappa \sum_p P(E_p | \sigma_j) P(E_p | \rho)^2 \\
&\quad - \alpha^3 \chi \eta P(E_i | \sigma_j) \sum_m P(E_m | \rho)^2 \\
&\quad - \alpha^2 \chi \beta \kappa \sum_m P(E_m | \rho)^2 \\
&\quad + \alpha \chi \kappa \gamma^{-1} (1 - n\eta) P(\rho | \rho) P(E_i | \sigma_j) \\
&\quad - \alpha \chi \beta (\alpha \eta + \kappa) P(E_i | \sigma_j) \\
&\quad + \kappa \chi (\kappa - \alpha \eta) P(E_i | \rho) \\
&\quad + \kappa \chi (\kappa - \alpha \eta) P(E_j | \rho) \\
&\quad \left. + \left[\kappa \chi \gamma^{-1} (2\kappa - n\beta \kappa - \alpha \beta) P(\rho | \rho) - 2\beta \chi \kappa^2 \right] \right\}.
\end{aligned}$$

and then

$$\begin{aligned}
& \left[\mathcal{L}[\rho^2] \Phi \right]_{ij} \\
&= \chi \left\{ \alpha^4 \chi \sum_m P(E_m | \sigma_i) Q(E_m | \sigma_j) \sum_p P(E_m | \sigma_p) P(E_p | \rho)^2 \right. \\
&\quad + \alpha^2 \chi (\kappa - \alpha \eta) \sum_m P(E_m | \sigma_i) Q(E_m | \sigma_j) P(E_m | \rho) \\
&\quad + \alpha^2 \chi \kappa \sum_p P(E_p | \sigma_i) P(E_p | \rho)^2 \\
&\quad + \alpha^2 \chi \kappa \sum_p Q(E_p | \sigma_j) P(E_p | \rho)^2 \\
&\quad - \alpha^3 \chi \eta Q(E_i | \sigma_j) \sum_m P(E_m | \rho)^2 \\
&\quad - \alpha^2 \chi \beta \kappa \sum_m P(E_m | \rho)^2 \\
&\quad + \alpha \chi \kappa \gamma^{-1} (1 - n\eta) P(\rho | \rho) Q(E_i | \sigma_j) \\
&\quad - \alpha \chi \beta (\alpha \eta + \kappa) Q(E_i | \sigma_j)
\end{aligned}$$

$$\begin{aligned}
& + \kappa\chi(\kappa - \alpha\eta)P(E_i|\rho) \\
& + \kappa\chi(\kappa - \alpha\eta)Q(E_j|\rho) \\
& + \left[\kappa\chi\gamma^{-1}(2\kappa - n\beta\kappa - \alpha\beta)P(\rho|\rho) - 2\beta\chi\kappa^2 \right] \Bigg\}.
\end{aligned}$$

And then it is a matter of calculation.