After some algebra...

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Let $\Phi = \alpha I + \beta J$ for $\beta = (1-\alpha)\left(\frac{1}{n}\right)$. Let

$$\begin{aligned} A_{ijk} &= \eta\Big(\delta_{ij} - \delta_{ik} - \delta_{jk}\ \Big) \\ B[Y]_{ij} &= \sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk}\Big) y_k \\ C[Y] &= P\Phi B[Y] P\Phi \\ \mathcal{L}[Y] &= \chi C[Y]. \end{aligned}$$

Then

$$\begin{aligned} B[Y]_{ij} &= \sum_k \Big(\delta_{ij}\delta_{ik} - \eta\Big[\delta_{ij} - \delta_{ik} - \delta_{jk}\ \Big]\Big) y_k \\ &= \delta_{ij} y_i - \delta_{ij}\eta\bar{y} + \eta y_i + \eta y_j, \end{aligned}$$

and

$$\mathcal{L}[Y]_{ij} = \chi\Bigg\{\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\sigma_j)y_m - \alpha\eta\bar{y}P(E_i|\sigma_j) + \kappa y_i + \kappa y_j - \beta\kappa\bar{y}\Bigg\},$$

where $\kappa = \alpha\eta + \beta + n\beta\eta = \beta + \eta$. Let $Q(E|\sigma) = \Phi P(E|\sigma)$ and $Q(E|\rho) = \Phi P(E|\rho)$. Moreover,

$$\Big[\mathcal{L}[\rho]\Phi\Big]_{ij} = \chi\Bigg\{\alpha^2 \sum_m P(E_m|\sigma_i)Q(E_m|\sigma_j)P(E_m|\rho) - \alpha\eta Q(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa Q(E_j|\rho) - \beta\kappa\Bigg\}.$$

Suppose we demand commutativity: $\mathcal{L}[\rho]\Phi P(E|\tau) = \mathcal{L}[\tau]\Phi P(E|\rho)$. We have

$$\begin{aligned} \mathcal{L}[\rho]\Phi P(E|\tau) &= \chi\Bigg\{\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\tau)P(E_m|\rho) - \alpha\eta P(E_i|\tau) + \kappa P(E_i|\rho) + \kappa\gamma^{-1}P(\rho|\tau) - \beta \\ \mathcal{L}[\tau]\Phi P(E|\rho) &= \chi\Bigg\{\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\rho)P(E_m|\tau) - \alpha\eta P(E_i|\rho) + \kappa P(E_i|\tau) + \kappa\gamma^{-1}P(\rho|\tau) - \beta \end{aligned}$$

from which we conclude that $\eta = -\frac{\beta}{\alpha+1}$. Let us further assume self-duality for our reference states and effects so that invoking Bayes' rule, we have

$$P(\rho|\sigma_i) = P(\rho|E_i) = \frac{P(E_i|\rho)P(\rho)}{P(E_i)} = \gamma P(E_i|\rho), \tag{1}$$

so that $\gamma P(E|\rho) = P(\rho|\sigma)$. If we assume self-duality in general, then

$$P(\rho|\rho) = P(\rho|\sigma)\Phi P(E|\rho) = \gamma P(E|\rho)\Phi P(E|\rho)$$
$$= \gamma\left[\alpha \sum_i P(E_i|\rho)^2 + \beta\right].$$

If $P(\rho|\rho) = 1$, then

$$\sum_i P(E_i|\rho)^2 = \frac{1}{\alpha}\left[\gamma^{-1} - \beta\right].$$

In 3-design quantum theory, $\alpha = d+1$, $\beta = -\frac{d}{n}$, and $\gamma = \frac{n}{d}$, which gives $\sum_i P(E_i|\rho)^2 = \left(\frac{d}{n}\right)\frac{2}{d+1}$.

Let us now consider $\tilde{P}(E|\rho^2) = \mathcal{L}[\rho]\Phi P(E|\rho)$, or

$$\tilde{P}(E_i|\rho^2) = \chi\left\{\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\rho)^2 + (\kappa - \alpha\eta)P(E_i|\rho) + \kappa\gamma^{-1}P(\rho|\rho) - \beta\kappa\right\}.$$

If we further demand that $\rho = \rho^2$, or $P(E_i|\rho) = \tilde{P}(E_i|\rho^2)$, then

$$P(E_i|\rho) = \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right)\left[\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta)\right],$$

and since $\sum_i P(E_i|\rho) = 1$, we have

$$\left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right)\left[\alpha^2 \sum_m P(E_m|\rho)^2 + n\kappa(\gamma^{-1} - \beta)\right] = 1,$$

or $\alpha(\gamma^{-1} - \beta) + n\kappa(\gamma^{-1} - \beta) = \chi^{-1} - \kappa + \alpha\eta$, which fixes

$$\chi = \frac{\gamma(\alpha+1)}{2\alpha}.$$

So by assuming **commutativity**, that is, $\mathcal{L}[\rho]\Phi P(E|\tau) = \mathcal{L}[\tau]\Phi P(E|\rho)$, we can fix η. By assuming **self-duality**, and for **pure states** that $P(\rho|\rho) = 1$ and $P(E|\rho) = \tilde{P}(E|\rho^2)$, we can fix $\sum_i P(E_i|\rho)^2$ and then χ. (Notice we haven't said anything about the extremal states as defined by the variance restriction alone.) Indeed, now

$$\mathcal{L}[Y]_{ij} = \frac{1}{2}\gamma\left\{\alpha(\alpha+1)\sum_m P(E_m|\sigma_i)P(E_m|\sigma_j)y_m - \left(\frac{\alpha-1}{n}\right)\left(\bar{y}P(E_i|\sigma_j) + y_i + y_j\right) - \left(\frac{\alpha-1}{n}\right)^2\right.$$

We can go further using $P(\rho|\rho) = \gamma P(E|\rho)\Phi P(E|\rho) = 1$ by substituting in the expression for $\tilde{P}(E|\rho^2)$. Thus

$$1 = \gamma \sum_i Q(E_i|\rho) \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right) \left[\alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta)\right]$$
$$= \left(\frac{\gamma}{\chi^{-1} - \kappa + \alpha\eta}\right) \left[\alpha^2 \sum_m P(E_m|\rho)^3 + \kappa(\gamma^{-1} - \beta)\right],$$

so that

$$\sum_m P(E_m|\rho)^3 = \frac{1}{\alpha^2} \left[\left(\frac{\chi^{-1} - \kappa + \alpha\eta}{\gamma}\right) - \kappa(\gamma^{-1} - \beta)\right]$$
$$= \frac{((\alpha-1)\gamma + n)((\alpha-1)\gamma + 2n)}{\alpha(\alpha+1)\gamma^2 n^2}.$$

In 3-design quantum theory, $\alpha = d + 1$ and $\gamma = \frac{n}{d}$, and so $\sum_i P(E_i|\rho)^3 = \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}$. Can we keep going? No! For example, we also have

$$P(\rho|\rho) = 1$$
$$= \sum_i \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right)^2 \left[\alpha^2 \sum_m P(E_m|\sigma_i) P(E_m|\rho)^2 + \kappa(\gamma^{-1} - \beta)\right] \left[\alpha^2 \sum_q Q(E_q|\sigma_i) P(E_q|\rho)^2 +$$
$$= \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right)^2 \left\{\alpha^4 \sum_{mq} P(E_m|\rho)^2 P(E_q|\rho)^2 P(E_m|\sigma_q) + 2\alpha^2\kappa(\gamma^{-1} - \beta) \sum_m P(E_m|\rho)^2 + n\kappa\right.$$
$$= \gamma \left(\frac{1}{\chi^{-1} - \kappa + \alpha\eta}\right)^2 \left\{\alpha^4 \sum_{mq} P(E_m|\rho)^2 P(E_q|\rho)^2 P(E_m|\sigma_q) + 2\kappa\alpha(\gamma^{-1} - \beta)^2 + n\kappa^2(\gamma^{-1} - \beta)^2\right\}$$

so that

$$\sum_{mq} P(E_m|\rho)^2 P(E_m|\sigma_q) P(E_q|\rho)^2 = \frac{1}{\alpha^4} \left\{\gamma^{-1}(\chi^{-1} - \kappa + \alpha\eta)^2 - 2\kappa\alpha(\gamma^{-1} - \beta)^2 - n\kappa^2(\gamma^{-1} - \beta)^2\right\}$$
$$= \frac{((\alpha-1)\gamma + n)^2((\alpha-1)(\alpha+3)\gamma + 4n)}{\alpha^2(\alpha+1)^2\gamma^3 n^3},$$

which for 3-design quantum theory becomes $\frac{4d^3(d+8)}{(d+1)^2(d+2)^2n^3}$, which is true! But the pure probability-assignments don't live on $(p > 3)$-norm spheres. (Does the 2-norm and 3-norm sphere condition imply that $P(E|\rho) = \tilde{P}(E|\rho^2)$?)

It would be nice to fix γ in terms of α. The next thing, however, to consider is the Jordan identity itself, $\mathcal{L}[\rho]\Phi = \mathcal{L}[\rho^2]\Phi$. On the one hand,

$$\Big[\mathcal{L}[\rho]\Phi\Big]_{ij} = \chi \left\{\alpha^2 \sum_m P(E_m|\sigma_i) Q(E_m|\sigma_j) P(E_m|\rho) - \alpha\eta Q(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa Q(E_j|\rho) - \beta\kappa\right\}.$$

On the other hand,

$$\tilde{P}(E_i|\rho^2) = \chi\left\{\alpha^2 \sum_m P(E_m|\sigma_i)P(E_m|\rho)^2 + (\kappa - \alpha\eta)P(E_i|\rho) + \kappa\gamma^{-1}P(\rho|\rho) - \beta\kappa\right\}$$

and

$$\sum_i \tilde{P}(E_i|\rho^2) = \chi\left\{\alpha^2 \sum_m P(E_m|\rho)^2 + n\kappa\gamma^{-1}P(\rho|\rho) + \alpha\beta\right\},$$

so that

$$\begin{aligned}
&\mathcal{L}[\rho^2]_{ij} \\
&= \chi\Bigg\{\alpha^4\chi \sum_m P(E_m|\sigma_i)P(E_m|\sigma_j) \sum_p P(E_m|\sigma_p)P(E_p|\rho)^2 \\
&+ \alpha^2\chi(\kappa - \alpha\eta) \sum_m P(E_m|\sigma_i)P(E_m|\sigma_j)P(E_m|\rho) \\
&+ \alpha^2\chi\kappa \sum_p P(E_p|\sigma_i)P(E_p|\rho)^2 \\
&+ \alpha^2\chi\kappa \sum_p P(E_p|\sigma_j)P(E_p|\rho)^2 \\
&- \alpha^3\chi\eta P(E_i|\sigma_j) \sum_m P(E_m|\rho)^2 \\
&- \alpha^2\chi\beta\kappa \sum_m P(E_m|\rho)^2 \\
&+ \alpha\chi\kappa\gamma^{-1}\big(1 - n\eta\big)P(\rho|\rho)P(E_i|\sigma_j) \\
&- \alpha\chi\beta\big(\alpha\eta + \kappa\big)P(E_i|\sigma_j) \\
&+ \kappa\chi(\kappa - \alpha\eta)P(E_i|\rho) \\
&+ \kappa\chi(\kappa - \alpha\eta)P(E_j|\rho) \\
&+ \Big[\kappa\chi\gamma^{-1}\big(2\kappa - n\beta\kappa - \alpha\beta\big)P(\rho|\rho) - 2\beta\chi\kappa^2\Big]\Bigg\}.
\end{aligned}$$

and then

$$
\begin{aligned}
&\Big[\mathcal{L}[\rho^2]\Phi\Big]_{ij} \\
&= \chi\Bigg\{\alpha^4\chi\sum_m P(E_m|\sigma_i)Q(E_m|\sigma_j)\sum_p P(E_m|\sigma_p)P(E_p|\rho)^2 \\
&+ \alpha^2\chi(\kappa-\alpha\eta)\sum_m P(E_m|\sigma_i)Q(E_m|\sigma_j)P(E_m|\rho) \\
&+ \alpha^2\chi\kappa\sum_p P(E_p|\sigma_i)P(E_p|\rho)^2 \\
&+ \alpha^2\chi\kappa\sum_p Q(E_p|\sigma_j)P(E_p|\rho)^2 \\
&- \alpha^3\chi\eta Q(E_i|\sigma_j)\sum_m P(E_m|\rho)^2 \\
&- \alpha^2\chi\beta\kappa\sum_m P(E_m|\rho)^2 \\
&+ \alpha\chi\kappa\gamma^{-1}\big(1-n\eta\big)P(\rho|\rho)Q(E_i|\sigma_j) \\
&- \alpha\chi\beta\big(\alpha\eta+\kappa\big)Q(E_i|\sigma_j) \\
&+ \kappa\chi(\kappa-\alpha\eta)P(E_i|\rho) \\
&+ \kappa\chi(\kappa-\alpha\eta)Q(E_j|\rho) \\
&+ \Big[\kappa\chi\gamma^{-1}\big(2\kappa-n\beta\kappa-\alpha\beta\big)P(\rho|\rho)-2\beta\chi\kappa^2\Big]\Bigg\}.
\end{aligned}
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And then it is a matter of calculation.