QBuki Notes on Reconstruction

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Procession

A (complex-projective, unbiased) t-design is a set of pure quantum states $\{\sigma_i\}_{i=1}^n$ which satisfy

$$\frac{1}{n}\sum_i \sigma_i^{\otimes t} = \int |\psi\rangle\langle\psi|^{\otimes t} d\psi = \binom{d+t-1}{t}^{-1} \Pi_{\mathrm{sym}^t}. \tag{1}$$

1-designs, rescaled, form measurements. 2-designs include SICs and MUBs. 3-designs will concern us here. Let us consider a measure-and-prepare device which performs measurement $\{E_i = \frac{d}{n}\sigma_i\}$ and conditionally prepares a state $\{\sigma_i\}$ where the states $\{\sigma_i\}$ form an 3-design. Note that a t-design is also a $(t-1)$-design, so that indeed, $\{E_i\}$ is a measurement. Moreover, from the fact that the device forms a 2-design, it is informationally-complete: the probabilities $P(E_i|\rho) = \mathrm{tr}(E_i\rho)$ suffice to pick out a density matrix. But crucially, not all probability distributions correspond to valid states (they map to matrices with negative eigenvalues). So how can we characterize the valid probability-assignments to the reference device? Here the 3-design property comes into play.

Consider the agreement-probabilities for t devices

$$\begin{aligned} P(\mathrm{agree}|\rho_1, \dots, \rho_t) &= \sum_{i=1}^n \prod_{j=1}^t P(E_i|\rho_j) = \mathrm{tr}\left(\sum_{i=1}^n E_i^{\otimes t} \otimes_{j=1}^t \rho_j\right) \\ &= \frac{d^t}{n^{t-1}}\binom{d+t-1}{t}^{-1}\frac{1}{t!}\sum_{\pi\in S_t}\mathrm{tr}(T_\pi \otimes_{j=1}^t \rho_j). \end{aligned} \tag{2}$$

To evaluate this, note that

$$\mathrm{tr}\left((X\otimes Y)\sum_{ab}|b,a\rangle\langle a,b|\right) = \sum_{ab}\langle a|X|b\rangle\langle b|Y|a\rangle = \mathrm{tr}(XY) \tag{3}$$

$$\mathrm{tr}\left((X\otimes Y\otimes Z)\sum_{abc}|b,c,a\rangle\langle a,b,c|\right) = \sum_{abc}\langle a|X|b\rangle\langle b|Y|c\rangle\langle c|Z|a\rangle = \mathrm{tr}(XYZ \tag{4}$$

On the one hand,

$$P(\text{agree}|\rho_1,\rho_2) = \frac{1}{d+1}\left(\frac{d}{n}\right)\Big[\text{tr}(\rho_1)\text{tr}(\rho_2) + \text{tr}(\rho_1\rho_2)\Big] \leq \left(\frac{d}{n}\right)\frac{2}{d+1}, \quad (5)$$

which is maximized when $\rho_1 = \rho_2 = \rho$ pure. On the other hand,

$$\begin{aligned} P(\text{agree}|\rho_1,\rho_2,\rho_3) = \frac{1}{(d+1)(d+2)}\left(\frac{d}{n}\right)^2 \Big[&\text{tr}(\rho_1)\text{tr}(\rho_2)\text{tr}(\rho_3) + \text{tr}(\rho_1)\text{tr}(\rho_2\rho_3) \\ &+ \text{tr}(\rho_2)\text{tr}(\rho_1\rho_3) + \text{tr}(\rho_3)\text{tr}(\rho_1\rho_2) + \text{tr}(\rho_1\rho_2\rho_3) + \text{tr}(\rho_1\rho_3\rho_2)\Big] \\ &\leq \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}, \end{aligned} \quad (6)$$

which again is maximized when $\rho_1 = \rho_2 = \rho_3 = \rho$ pure. Consider the following lemma:

Lemma 1. *A quantum state ρ is pure if and only if $tr(\rho^2) = tr(\rho^3) = 1$.*

Proof. Let $\{\lambda_i\}$ be the eigenvalues of ρ. $\text{tr}(\rho^2) = \text{tr}(\rho^3) = 1$ means that $\sum_i \lambda_i^2 = \sum_i \lambda^3 = 1$. On the one hand, $\sum_i \lambda_i^2 = 1$ implies that $\forall i : -1 \leq \lambda_i \leq 1$. On the other hand, $\sum_i \lambda_i^3 \leq \sum_i \lambda_i^2$ with equality if and only if $\forall i : \lambda_i \in \{0,1\}$. But since the whole sum must be 1, we must have exactly one $\lambda_i = 1$ and the rest 0. Thus ρ is a rank-1 projector, and hence a pure state. □

We conclude that we can characterize pure-states by the following equations

$$\forall i : P(E_i|\rho) \geq 0 \quad (7)$$

$$\sum_i P(E_i|\rho) = 1 \quad (8)$$

$$\sum_i P(E_i|\rho)^2 = \left(\frac{d}{n}\right)\frac{2}{d+1} \quad (9)$$

$$\sum_i P(E_i|\rho)^3 = \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}, \quad (10)$$

with the caveat that $P(E|\rho) \in \text{col}(P)$, where $P_{ij} = P(E_i|\sigma_j) = \text{tr}(E_i\sigma_j)$. Why this last condition? The reason is that a 3-design representation is necessarily overcomplete—indeed, $n \geq \frac{1}{2}d^2(d+1)$— and in our derivation, we've assumed that all probabilities $P(E_i|\rho)$ are obtained from $\text{tr}(E_i\rho)$. Let $\mathbf{E}$ be the matrix whose rows are $(E_i|$ and $\mathbf{S}$ be the matrix whose columns are $|\sigma_i)$ where $|X) = (X \otimes I)\sum_i |i,i\rangle = \text{vec}(X)$. On the one hand, $P(E|\rho) = \mathbf{E}|\rho)$; on the other hand, $P = \mathbf{ES}$, which is a full-rank factorization and thus the columns of $\mathbf{E}$ form a basis for the column-space of P. Therefore our proof becomes if-and-only if as long as $P(E|\rho) \in \text{col}(P)$.

It is worth noting that we can motivate the restriction that $P(E|\rho) \in \mathrm{col}(P)$ on QBist grounds. Let the Born matrix Φ be any matrix satisfying $P\Phi P = P \iff \mathbf{S}\Phi\mathbf{E} = I$. Then the Born rule appears as

$$P(A_i|\rho) = \mathrm{tr}(A_i\rho) = (A_i|\mathbf{S}\Phi\mathbf{E}|\rho) = \sum_{jk} P(A_i|\sigma_j)\Phi_{jk}P(E_k|\rho), \tag{11}$$

a deformation of the law of total probability. In particular, $P(E_i|\rho) = \sum_{jk} P(E_i|\sigma_j)\Phi_{jk}P(E_k|\rho)$. Thus for consistency's stake, we ought to require $P(E|\rho) = P\Phi P(E|\rho)$. $P\Phi P = P$ implies that $P\Phi$ is a projector. On what subspace, though? For a 2-design $\frac{1}{n}\sum_i \sigma_i^{\otimes 2} = \frac{1}{d(d+1)}(I \otimes I + \mathcal{S})$ so that

$$\frac{1}{n}\sum_i \sigma_i \otimes \sigma_i^T = \frac{1}{d(d+1)}\left(I \otimes I + |I)(I|\right). \tag{12}$$

For a pure state σ, $|\sigma)(\sigma| = \sigma \otimes \sigma^T$, and so letting $E_i = \frac{d}{n}\sigma_i$, we arrive at the resolution of the identity $I = (d+1)\sum_i |\sigma_i)(E_i| - |I)(I|$, which demonstrates informational-completeness. Comparing this to $\mathbf{S}\Phi\mathbf{E} = I = \sum_{ij} \Phi_{ij}|\sigma_i)(E_j|$, it follows that we may take $\Phi = (d+1)I - \frac{d}{n}J$. Since Φ is full rank, $P\Phi$ projects onto $\mathrm{col}(P)$.

So the contour of quantum state-space according to a 3-design is given by the intersection of the non-negative orthant, a 1-norm sphere, a 2-norm sphere, and a 3-norm sphere of prescribed radii, and a d^2 dimensional subspace: $\mathrm{col}(P)$. Alternatively, we can derive a single equation picking out pure probability-assignments from the demand that $\rho = \rho^2$. From the resolution of the identity, $\rho = \sum_{ij} \Phi_{ij}P(E_j|\rho)\sigma_i$, we have

$$P(E_i|\rho) = \sum_{km} P(E_k|\rho)P(E_m|\rho)\sum_{jm} \Phi_{jk}\Phi_{lm}\Re\big[\mathrm{tr}(E_i\sigma_j\sigma_l)\big], \tag{13}$$

where the real-part comes from $\mathrm{tr}(E_i\sigma_j\sigma_l) + \mathrm{tr}(E_i\sigma_l\sigma_j) = \mathrm{tr}(E_i\sigma_j\sigma_l) + \mathrm{tr}(E_i\sigma_j\sigma_l)^* = 2\Re[\mathrm{tr}(E_i\sigma_j\sigma_l)]$. Let $\mathcal{M}_3 = \frac{1}{n}\sum_i \sigma_i^{\otimes 3}$ so that

$$\begin{aligned} P(E_i, E_j, E_k|\mathcal{M}_3) &= \frac{1}{n}\sum_m P(E_i|\sigma_m)P(E_j|\sigma_m)P(E_k|\sigma_m) \qquad (14) \\ &= \frac{1}{(d+1)(d+2)}\left(\frac{d}{n^2}\right)\Bigg[\frac{d}{n} + P(E_j|\sigma_k) + P(E_i|\sigma_j) + P(E_i|\sigma_k) + 2\Re\big[\mathrm{tr}(E_i\sigma_j\sigma_k \end{aligned}$$

and therefore

$$\begin{aligned}&\Re\big[\mathrm{tr}(E_i\sigma_j\sigma_k)\big] &(15)\\ &=\frac{1}{2}\Bigg[(d+1)(d+2)\left(\frac{n}{d}\right)\sum_m P(E_i|\sigma_m)P(E_j|\sigma_m)P(E_k|\sigma_m)-P(E_j|\sigma_k)-P(E_i|\sigma_j)-P(E_i|\sigma_k)-\end{aligned}$$

Our condition for pure-statehood then simplifies to

$$P(E_i|\rho)=\frac{1}{2}\left[\frac{1}{2}(d+1)(d+2)\left(\frac{n}{d}\right)\sum_m P(E_i|\sigma_m)P(E_m|\rho)^2-\frac{d}{n}\right], \tag{16}$$

which we note automatically implies $P(E|\rho)\in\mathrm{col}(P)$.

In fact, we can do even better, and derive a condition for the validity of any state, pure or mixed. We note that $\Re\big[\mathrm{tr}(E_i\sigma_j\sigma_k)\big]$ are essentially the structure-coefficients for the Jordan product $A\circ B=\frac{1}{2}(AB+BA)$,

$$\mathrm{tr}(E_iA\circ B)=\frac{1}{2}\big(\mathrm{tr}(E_iAB)+\mathrm{tr}(E_iBA)\big)=\sum_{km}\mathrm{tr}(E_kA)\mathrm{tr}(E_mB)\sum_{jl}\Phi_{jk}\Phi_{lm}\Re \tag{17}$$

The linear operator $L[\rho]$ which performs the Jordan product (and which acts on vectorized states) is $L[\rho]=\frac{1}{2}(\rho\otimes I+I\otimes\rho^T)$. The matrix

$$\begin{aligned}\mathcal{L}[\rho]_{ij}=\mathrm{tr}(E_iL[\rho](\sigma_j))&=\frac{d}{n}(\sigma_i|L[\rho]|\sigma_j) &(18)\\ &=\sum_{kl}\Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]\Phi_{kl}P(E_l|\rho) &(19)\end{aligned}$$

does the same on e.g. probability vectors:

$$\tilde{P}(E|\rho\circ\tau)=\mathbf{E}L[\rho]|\tau)=\mathbf{E}L[\rho]\mathbf{S}\Phi\mathbf{E}|\tau)=\mathcal{L}[\rho]\Phi P(E|\tau), \tag{20}$$

where the tilde recalls that $\tilde{P}(E|\rho\circ\tau)$ might not be a normalized probability distribution. Note that $\mathcal{L}[\rho]$ does not depend on any redundancy in $P(E|\rho)$. We find after substitution that

$$\mathcal{L}[\rho]_{ij}=\frac{1}{2}\Bigg[(d+1)(d+2)\left(\frac{n}{d}\right)\sum_m P(E_m|\sigma_i)P(E_m|\sigma_j)P(E_m|\rho)-P(E_i|\sigma_j)- \tag{21}$$

Now clearly, $\rho\geq 0\iff L[\rho]\geq 0$. Moreover, $L[\rho]\geq 0\iff\mathcal{L}[\rho]\geq 0$. To see this note that if $\{|f_i)\}$ is a frame with dual elements $\{(\tilde{f}_i|\}$ such that $\sum_i|f_i)(\tilde{f}_i|=\sum_i|\tilde{f}_i)(f_i|=I$, we can write an arbitrary operator $A=\sum_{ij}(f_i|A|f_j)\ |\tilde{f}_i)(\tilde{f}_j|$, where, considering the matrix of coefficients

$A^f_{ij} = (f_i|A|f_j)$, $A^f \geq 0$ iff $A \geq 0$, since $\sum_{ij} x_i^*(f_i|A|f_j)x_j = y^\dagger A y \geq 0$. We thus have a condition for statehood, pure or mixed: $\mathcal{L}[\rho] \geq 0$, which again only depends on reference device probabilities.

Finally, let us give an interpretation of this last result. Consider that if $X = \sum_i x_i E_i$ is some arbitrary observable, it follows that

$$\forall X : \mathrm{tr}(X^2\rho) = \left(\frac{d}{n}\right) \sum_{ijkl} x_i x_j \Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]\Phi_{kl}P(E_l|\rho) \geq 0 \iff \rho \geq 0, \tag{22}$$

which is immediately equivalent to $\mathcal{L}[\rho] \geq 0$. Again substituting in $\Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]$ yields a condition on valid $P(E|\rho)$

$$\forall\{x_i\} : \sum_i \left(\sum_j P(E_i|\sigma_j)x_j\right)^2 P(E_i|\rho) \geq \frac{d}{(d+1)(d+2)} \left[\frac{1}{n} \sum_{ij} x_i P(E_i|\sigma_j)x_j\right. \tag{23}$$

where e.g. $\langle X|\rho\rangle = \sum_i x_i P(E_i|\rho)$, and $\forall i : P(E_i|\mu) = \frac{1}{n}$. If we make the simplifying assumption that $x \in \mathrm{col}(P)$, using the 2-design property, this simplifies to

$$\forall\{x_i\} \in \mathrm{col}(P) : \sum_i x_i^2 P(E_i|\rho) \geq \frac{d}{d+2}\Big(\langle X^2|\mu\rangle - 2\langle X|\mu\rangle\langle X|\rho\rangle\Big), \tag{24}$$

where e.g. $\langle X^2|\mu\rangle = \frac{1}{n}\sum_i x_i^2$. Notice that we are considering the second-moment *with respect to the reference device* as opposed to a von Neumann measurement (although the inequality is saturated iff $\mathrm{tr}(X^2\rho) = 0$). Thus the shape of quantum state-space can be understood in terms of a kind of uncertainty principle: a valid probability-assignment to the reference device implies a certain minimum variance to any observable in $\mathrm{col}(P)$.

Reversion

We begin in a formless void without yet quantum mechanics.

$\textbf{Assumption 1}$: There is a reference device characterized by a stochastic matrix $P_{ij} = P(E_i|\sigma_j)$ where P is symmetric and hence bistochastic.

$\textbf{Assumption 2}$: We assume that $\Phi = \alpha I + \beta J$ is a Born matrix for P, satisfying $P\Phi P = P$, and that $Q(E|\rho) = \Phi P(E|\rho)$ are quasi-probabilities, possibly negative, summing to 1. Here J is the matrix of all 1’s.

On the one hand, since $\sum_{ij} \Phi_{ij}P(E_j|\rho) = 1$, we must have $\alpha + n\beta = 1$ so that $\beta = (1-\alpha)\left(\frac{1}{n}\right)$. On the other hand,

$$P\Phi P = \alpha P^2 + \beta J = P. \tag{25}$$

Noting that $JPx = Jx$, we have $\alpha P(Px) + \beta J(Px) = Px$. Letting $y = Px \in \mathrm{col}(P)$ and $u = (1, \ldots, 1)^T$, we have

$$\alpha Py + \beta \left(\sum_i y_i \right) u = y \Longrightarrow Py = \frac{1}{\alpha} \left[y - \beta \left(\sum_i y_i \right) u \right]. \tag{26}$$

In particular, for probabilities $P(E|\rho) \in \mathrm{col}(P)$,

$$\sum_j P(E_i|\sigma_j) P(E_j|\rho) = \frac{1}{\alpha} P(E_i|\rho) + \left(1 - \frac{1}{\alpha} \right) \frac{1}{n} : \tag{27}$$

in other words, for probability-assignments in its column space, P acts as a depolarizing channel. We note that $P\Phi$ projects onto $\mathrm{col}(P)$.

Assumption 3: A probability-assignment $P(E|\rho)$ is valid if and only if for any observable $x \in \mathrm{col}(P)$, the second-moment with respect to the reference device satisfies a lower bound. Further we assume that like the second-moment itself, the lower bound is linear in $P(E|\rho)$ and quadratic in x.

We can characterize the lower bound in terms of a three-index tensor A_{ijk} such that a valid $P(E|\rho)$ satisfies

$$\forall \{x_i\} \in \mathrm{col}(P) : \sum_i x_i^2 P(E_i|\rho) \geq \sum_{ijk} A_{ijk} x_i x_j P(E_k|\rho), \tag{28}$$

or

$$\forall \{x_i\} \in \mathrm{col}(P) : \sum_{ij} x_i \left[\sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk} \Big) P(E_k|\rho) \right] x_j \geq 0. \tag{29}$$

Let $B[\rho]_{ij} = \sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk} \Big) P(E_k|\rho)$. Since $B[\rho] \geq 0$ on $\mathrm{col}(P)$, and $P\Phi = (P\Phi)^T$ projects onto that subspace, we have

$$C[\rho] = P\Phi B[\rho] P\Phi \geq 0 \tag{30}$$

simplicter iff $P(E|\rho)$ is a valid state. Indeed, if we choose A_{ijk} to be symmetric in the first two indices, then $C[\rho]$ will be postive semi-definite. We've thus managed to translate the validity of $P(E|\rho)$, expressed in terms of a lower bound on the second-moment of any observable with respect to the reference device, into the postive-semidefiniteness of a certain matrix associated to $P(E|\rho)$.

Assumption 4: We assume that $A_{ijk} = \eta \Big(\delta_{ij} - \delta_{ik} - \delta_{jk} \Big)$.

Substituting this simple form for A_{ijk} into the expression for $C[\rho]$ yields

$$C[\rho]_{ij} = \alpha^2 \sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) - \alpha\eta P(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa P(E \tag{31}$$

where $\kappa = \beta + \eta$.

Now let $\alpha = (d+1)$, $\beta = -\frac{d}{n}$, $\eta = \frac{1}{d+2}\left(\frac{d}{n}\right)$, and $\chi = \frac{1}{2}\left(\frac{n}{d}\right)\left(\frac{d+2}{d+1}\right)$. Then

$$\begin{aligned} \mathcal{L}[\rho]_{ij} &= \chi C[\rho]_{ij} \qquad (32) \\ &= \frac{1}{2}\Bigg\{(d+1)(d+2)\left(\frac{n}{d}\right)\sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) - P(E_i|\sigma_l) - P(E_i|\rho) - P(E_j|\rho) - \end{aligned}$$

is precisely the matrix we derived earlier, which represents taking the Jordan product with ρ. In other words, if $P(E_i|\sigma_j)$ in fact characterizes a quantum 3-design, then $\mathcal{L}[\rho]_{ij} = \mathrm{tr}\left(E_i\left[\frac{1}{2}(\sigma_j\rho + \rho\sigma_j)\right]\right)$.

Next steps:

- The Jordan product is completely characterized by its commutativity and the condition that $\left[L[\rho], L[\rho^2]\right] = 0$. For us, this means on the one hand, $\mathcal{L}[\rho]\Phi P(E|\tau) = \mathcal{L}[\tau]\Phi P(E|\rho)$, and on the other hand, $\left[\mathcal{L}[\rho]\Phi, \mathcal{L}[\rho^2]\Phi\right] = 0$. (Moreover, a *Euclidean* Jordan algebra satisfies $\forall A, B, C \in \mathcal{V} : \langle L[A]B, C\rangle = \langle B, L[A]C\rangle$ for a choice of inner product on the underlying vector space $\mathcal{V}$.) Does any

$$\mathcal{L}[\rho]_{ij} = \chi\Bigg\{\alpha^2 \sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) - \alpha\eta P(E_i|\sigma_j) + \kappa P(E_i|\rho) + \kappa P \tag{33}$$

  for arbitrary symmetric, stochastic, depolarizing $P(E|\sigma)$, given the appropriate choices of constants, satisfy the Jordan product conditions? In other words, have we found an alternative way of characterizing (some subset of) the Euclidean Jordan algebras? A great deal of tedious algebra lies in between resolving this yes or no. Suppose the answer is yes. Recall that all EJA's are direct sums of the simple EJA's: $\mathrm{Sym}(d, \mathbb{R})$, $\mathrm{Herm}(d, \mathbb{C})$, $\mathrm{Herm}(d, \mathbb{H})$, $\mathrm{Herm}(3, \mathbb{O})$, and $\mathbb{R} \times \mathbb{R}^{d-1}$. Then the choice of quantum theory over $\mathbb{C}$ is likely no simpler than the condition that $P(E_i|\sigma_j)$ can be represented as $\mathrm{tr}(E_i\sigma_j)$ for a 3-design $\{\sigma_i\}$ in $\mathcal{H}_d$. On the other hand, suppose the answer is no. Then the two defining conditions on the Jordan product translate into restrictions on the probabilities $P(E_i|\sigma_j)$. This could pick out a whole class of EJA's. Or if we're unreasonably lucky, it might pick out quantum theory over $\mathbb{C}$ specifically, and thus providing a characterization of 3-designs themselves entirely in terms of reference device probabilities $P(E_i|\sigma_j)$.

- Suppose instead we want to derive the Jordan structure. Answering the aforementioned question will likely suggest the best way of doing that. But we can already ask, for example: given some A_{ijk} alone (taking the simple form or not), a) can we characterize the extremal probability distributions? b) can we characterize its dual (the space of non-negative linear functionals of the form $P(\eta|\sigma)\Phi$)? Must such a state space be self-dual? Can we then show that iff $P(E|\rho)$ is extreme $\chi C[\rho]\Phi P(E|\rho) = P(E|\rho)$, for instance?