A 2-norm sphere

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We suppose self-duality and invoke Bayes' rule,

$$P(\rho|\sigma_i) = P(\rho|E_i) = \frac{P(E_i|\rho)P(\rho)}{P(E_i)} = \gamma P(E_i|\rho), \tag{1}$$

so that $\gamma P(E|\rho) = P(\rho|\sigma)$. We supposed that $\Phi = \alpha I + \beta J$ is a 1-inverse of P. Then

$$P(\eta|\rho) = P(\eta|\sigma)\Phi P(E|\rho) = \alpha P(\eta|\sigma)P(E|\rho) + \beta P(\eta|\sigma)u \tag{2}$$

$$= \alpha \sum_i P(\eta|\sigma_i)P(E_i|\rho) + \beta \sum_i P(\eta|\sigma_i). \tag{3}$$

Suppose we consider in particular

$$P(\rho|\rho) = P(\rho|\sigma)\Phi P(E|\rho) = \gamma P(E|\rho)\Phi P(E|\rho) \tag{4}$$

$$= \gamma \left[\alpha \sum_i P(E_i|\rho)^2 + \beta\right], \tag{5}$$

and then demand $P(\rho|\rho) = 1$. Thus

$$\sum_i P(E_i|\rho)^2 = \frac{1}{\alpha}\left[\frac{1}{\gamma} - \beta\right]. \tag{6}$$

In 3-design quantum theory, $\alpha = d + 1$, $\beta = -\frac{d}{n}$, and $\gamma = \frac{n}{d}$, which gives

$$\sum_i P(E_i|\rho)^2 = \left(\frac{d}{n}\right)\frac{2}{d+1}, \tag{7}$$

which we've derived elsewhere.