Some constants

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Let's forget everything again. We have a reference device characterized by some bistochastic $P(E_i|\sigma_j)$. We suppose that $\Phi = \alpha I + \beta J$ is a Born matrix, satisfying $P\Phi P = P$, where J is the matrix of all 1's. As we've learned this implies that for $P(E|\rho) \in \text{col}(P)$,

$$\sum_j P(E_i|\sigma_j)P(E_j|\rho) = \frac{1}{\alpha}\Big[P(E_i|\rho) - \beta\Big]. \tag{1}$$

If we want $\sum_{ij} \Phi_{ij} P(E_j) = 1$, we must have $\beta = (1-\alpha)\left(\frac{1}{n}\right)$. We observe that $P\Phi = (P\Phi)^T$ and $P\Phi = \alpha P + \beta J$, which projects on $S = \text{col}(P)$. For any observable $x \in \text{col}(P)$, we assume a lower bound on the variance which is linear in $P(E|\rho)$ and quadratic in $\{x_i\}$. We thus have some three-index tensor A_{ijk} such that

$$\forall x \in S : \sum_i x_i^2 P(E_i|\rho) \geq \sum_{ijk} A_{ijk} x_i x_j P(E_k|\rho), \tag{2}$$

or

$$\forall x \in S : \sum_{ij} x_i \left[\sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk}\Big) P(E_k|\rho)\right] x_j \geq 0. \tag{3}$$

Let $B[\rho]_{ij} = \sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk}\Big) P(E_k|\rho)$. We want A_{ijk} to be symmetric in the first two indices so that $B[\rho]$ is symmetric. Since $B[\rho] \geq 0$ on $\text{col}(P)$, and $P\Phi$ projects onto that subspace, we have

$$C[\rho] = \chi P\Phi B[\rho] P\Phi \geq 0 \tag{4}$$

iff $P(E|\rho)$ is a valid state. Just as we assumed a particular simple form for Φ, we assume a particular simple form for A_{ijk}, namely

$$A_{ijk} = \eta\Big(\delta_{ij} - \delta_{ik} - \delta_{jk}\Big). \tag{5}$$

Let us work out the matrix elements of $C[\rho]$. First,

$$\sum_k \Big(\delta_{ij}\delta_{ik} - \eta\Big[\delta_{ij} - \delta_{ik} - \delta_{jk}\ \Big]\Big) P(E_k|\rho) = \delta_{ij}P(E_i|\rho) - \eta\delta_{ij} + \eta P(E_i|\rho) + \eta. \tag{6}$$

so that

$$\begin{aligned}
C[\rho]_{il} &= \chi\Bigg\{\sum_{jk}\Big[\alpha P(E_i|\sigma_j) + \beta\Big]\Big[\delta_{jk}P(E_j|\rho) - \eta\delta_{jk} + \eta P(E_j|\rho) + \eta P(E_k|\rho)\Big]\Big[\alpha P(E_k|\sigma_l) + \beta\Big]\Bigg\} \\
&= \chi\Bigg\{\sum_{j}\Big[\alpha P(E_i|\sigma_j) + \beta\Big]\Big[\alpha P(E_j|\sigma_l)P(E_j|\rho) - \eta\alpha P(E_j|\sigma_l) \\
&\qquad + \eta\alpha P(E_j|\rho) + \eta\alpha\sum_k P(E_l|\sigma_k)P(E_k|\rho) \\
&\qquad + \beta P(E_j|\rho) - \eta\beta + n\eta\beta P(E_j|\rho) + \eta\beta\Big]\Bigg\} \\
&= \chi\Bigg\{\sum_{j}\Big[\alpha P(E_i|\sigma_j) + \beta\Big]\Big[\alpha P(E_j|\sigma_l)P(E_j|\rho) - \eta\alpha P(E_j|\sigma_l) \\
&\qquad + \eta P(E_l|\rho) + (\eta\alpha + \beta + n\eta\beta)P(E_j|\rho) - \beta\eta\Big]\Bigg\},
\end{aligned} \tag{7}$$

$$\begin{aligned}
&= \chi\Bigg\{\alpha^2\sum_j P(E_j|\sigma_i)P(E_j|\sigma_l)P(E_j|\rho) - \alpha^2\eta\sum_j P(E_i|\sigma_j)P(E_j|\sigma_l) \\
&\quad + \alpha\eta P(E_l|\rho) + \alpha(\eta\alpha + \beta + n\eta\beta)\sum_j P(E_i|\sigma_j)P(E_j|\rho) - \alpha\beta\eta \\
&\quad + \alpha\beta\sum_j P(E_l|\sigma_j)P(E_j|\rho) - \eta\alpha\beta + \beta\eta n P(E_l|\rho) + \beta(\eta\alpha + \beta + n\eta\beta) - \beta^2\eta n\Bigg\},
\end{aligned} \tag{8}$$

and since $\sum_j P(E_i|\sigma_j)P(E_j|\rho) = \frac{1}{\alpha}\Big[P(E_i|\rho) - \beta\Big]$, this simplifies to

$$\begin{aligned}
C[\rho]_{il} &= \chi\Bigg\{\alpha^2\sum_j P(E_j|\sigma_i)P(E_j|\sigma_l)P(E_j|\rho) - \alpha\eta P(E_i|\sigma_l) + \alpha\eta\beta + \alpha\eta P(E_l|\rho) \\
&+ (\eta\alpha + \beta + n\eta\beta)P(E_i|\rho) - \beta(\eta\alpha + \beta + n\eta\beta) - \alpha\beta\eta \\
&+ \beta P(E_l|\rho) - \beta^2 - \eta\alpha\beta + \beta\eta n P(E_l|\rho) + \beta(\eta\alpha + \beta + n\eta\beta) - \beta^2\eta n\Bigg\},
\end{aligned} \tag{9}$$

or

$$C[\rho]_{il} = \chi\Bigg\{\alpha^2 \sum_j P(E_j|\sigma_i)P(E_j|\sigma_l)P(E_j|\rho) - \alpha\eta P(E_i|\sigma_l) \tag{10}$$
$$+ (\eta\alpha + \beta + n\eta\beta)P(E_i|\rho) + (\eta\alpha + \beta + n\eta\beta)P(E_l|\rho)$$
$$- \beta(\eta\alpha + \beta + n\eta\beta)\Bigg\}.$$

Let $\kappa = \eta\alpha + \beta + n\eta\beta = \beta + \eta$, since $\beta = \frac{1}{n}(1-\alpha)$. We obtain at last

$$C[\rho]_{ij} = \chi\Bigg\{\alpha^2 \sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) - \alpha\eta P(E_i|\sigma_j) \tag{11}$$
$$+ \kappa P(E_i|\rho) + \kappa P(E_j|\rho) - \beta\kappa\Bigg\}.$$

In the case of quantum theory according to a 3-design, $\alpha = (d+1)$, $\beta = -\frac{d}{n}$, $\eta = \frac{1}{d+2}\left(\frac{d}{n}\right)$, $\chi = \frac{1}{2}\left(\frac{n}{d}\right)\left(\frac{d+2}{d+1}\right)$. Then $\kappa = -\left(\frac{d}{n}\right)\left(\frac{d+1}{d+2}\right) = -\alpha\eta$, and so

$$C[\rho]_{ij} = \frac{1}{2}\left(\frac{n}{d}\right)\left(\frac{d+2}{d+1}\right)\Bigg\{(d+1)^2 \sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) - \left(\frac{d}{n}\right)\left(\frac{d+1}{d+2}\right)P(E_i|\sigma_l)$$
$$- \left(\frac{d}{n}\right)\left(\frac{d+1}{d+2}\right)P(E_i|\rho) - \left(\frac{d}{n}\right)\left(\frac{d+1}{d+2}\right)P(E_j|\rho) - \left(\frac{d}{n}\right)^2\left(\frac{d+1}{d+2}\right)\Bigg\}, \tag{12}$$

or

$$C[\rho]_{ij} = \frac{1}{2}\Bigg\{(d+1)(d+2)\left(\frac{n}{d}\right)\sum_k P(E_k|\sigma_i)P(E_k|\sigma_j)P(E_k|\rho) \tag{13}$$
$$- P(E_i|\sigma_l) - P(E_i|\rho) - P(E_j|\rho) - \frac{d}{n}\Bigg\},$$

which is precisely the matrix $\mathcal{L}[\rho]_{ij} = \mathrm{tr}\left(E_i\left[\frac{1}{2}(\sigma_j\rho + \rho\sigma_j)\right]\right)$ which we've derived elsewhere, and in particular $C[\sigma_k]_{ij} = \Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]$, so that $C[\rho] = \sum_k C[\sigma_k]\Phi_{kl}P(E_l|\rho)$.