Depolarization and the Urgleichung

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We've supposed that $\Phi = \alpha I + \beta J$ is a 1-inverse of P. The constants may be related by demanding that Φ preserve the 1-norm. Let u be the vector of all 1's. Then

$$u^T \Phi x = \alpha u^T x + \beta u^T J x = (\alpha + \beta n) u^T x. \tag{1}$$

Thus $\beta = (1-\alpha)\left(\frac{1}{n}\right)$. Now a 1-inverse of P satisfies $P\Phi P = P$. What does imposing this form for Φ imply about P? We have $P\Big[\alpha I + \beta J\Big]P = \alpha P^2 + \beta J = P$, so that

$$P^2 = \frac{1}{\alpha}\Big[P - \beta J\Big] \tag{2}$$

$$P(Px) = \frac{1}{\alpha}\Big[(Px) - \beta J(Px)\Big] \tag{3}$$

$$Py = \frac{1}{\alpha}\Big[y - \beta J y\Big], \tag{4}$$

where $y \in \text{col}(P)$, and we note that since we've assumed P symmetric and stochastic (bistochastic) $JPx = Jy$. Thus for $y \in \text{col}(P)$,

$$\sum_j P(E_i|\sigma_j) y_j = \frac{1}{\alpha}\left[y_i - \beta \sum_j y_j\right]. \tag{5}$$

In particular, for $P(E|\rho) \in \text{col}(P)$,

$$\sum_j P(E_i|\sigma_j) P(E_j|\rho) = \frac{1}{\alpha}\Big[P(E_i|\rho) - \beta\Big] \tag{6}$$

$$= \frac{1}{\alpha} P(E_i|\rho) + \left(1 - \frac{1}{\alpha}\right)\frac{1}{n}, \tag{7}$$

which has the form of a depolarizing channel with strength $\frac{1}{\alpha}$. Thus on the privileged subspace $S = \text{col}(P)$, the measure-and-prepare channel associated with the reference device simply mixes with the flat probability vector. (And if a vector has any component in $S^\perp$, this part is projected away.) Geometrically, the channel performs an isotropic contraction of the state-space. Indeed, this is a property shared by quantum 2-designs.