# Relating radii

written by heyredhat on Functor Network
original link: https://functor.network/user/1704/entry/649

---

Last time we learned that in quantum theory only for a pure state does $O = I - \rho$ saturate the lower-bound

$$\langle O^2|\rho\rangle \geq \frac{d}{d+2}\Big[\langle O^2|\mu\rangle - 2\langle O|\mu\rangle\langle O|\rho\rangle\Big]. \tag{1}$$

We have $\mathrm{tr}(O\sigma_i) = 1 - \mathrm{tr}(\rho\sigma_i)$, and so $o_i = 2 - \left(\frac{n}{d}\right)(d+1)P(E_i|\rho)$. From the LHS, we have

$$\langle O^2|\rho\rangle = \sum_i \left[4 - 4\left(\frac{n}{d}\right)(d+1)P(E_i|\rho) + \left(\frac{n}{d}\right)^2 (d+1)^2 P(E_i|\rho)^2\right] P(E_i|\rho) \tag{2}$$

$$= 4 - 4\left(\frac{n}{d}\right)(d+1)\sum_i P(E_i|\rho)^2 + \left(\frac{n}{d}\right)^2 (d+1)^2 \sum_i P(E_i|\rho)^3. \tag{3}$$

From the right hand side, we have

$$\langle O^2|\rho\rangle = \frac{d}{d+2}\left[4 - 4\left(\frac{1}{d}\right)(d+1) + \left(\frac{n}{d^2}\right)(d+1)^2 \sum_i P(E_i|\rho)^2\right]. \tag{4}$$

Equating the two implies that

$$\sum_i P(E_i|\rho)^3 = \left(\frac{d^2}{n^2}\right)\left(\frac{1}{d(d+1)^2(d+2)}\right) \times \left[n(5d^2+14d+9)\sum_i P(E_i|\rho)^2 - 4d^2 - 12d\right], \tag{5}$$

which indeed relates

$$\sum_i P(E_i|\rho)^2 = \left(\frac{d}{n}\right)\frac{2}{d+1} \tag{6}$$

$$\sum_i P(E_i|\rho)^3 = \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}, \tag{7}$$

which I derive in the paper, and is equivalent to $\mathrm{tr}(\rho^2) = \mathrm{tr}(\rho^3) = 1$. It turns out that this is enough to pick out pure $\rho$.