

Relating radii

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Last time we learned that in quantum theory only for a pure state does $O = I - \rho$ saturate the lower-bound

$$\langle O^2 | \rho \rangle \geq \frac{d}{d+2} \left[\langle O^2 | \mu \rangle - 2 \langle O | \mu \rangle \langle O | \rho \rangle \right]. \quad (1)$$

We have $\text{tr}(O\sigma_i) = 1 - \text{tr}(\rho\sigma_i)$, and so $o_i = 2 - \left(\frac{n}{d}\right) (d+1)P(E_i|\rho)$. From the LHS, we have

$$\langle O^2 | \rho \rangle = \sum_i \left[4 - 4 \left(\frac{n}{d}\right) (d+1)P(E_i|\rho) + \left(\frac{n}{d}\right)^2 (d+1)^2 P(E_i|\rho)^2 \right] P(E_i|\rho) \quad (2)$$

$$= 4 - 4 \left(\frac{n}{d}\right) (d+1) \sum_i P(E_i|\rho)^2 + \left(\frac{n}{d}\right)^2 (d+1)^2 \sum_i P(E_i|\rho)^3. \quad (3)$$

From the right hand side, we have

$$\langle O^2 | \rho \rangle = \frac{d}{d+2} \left[4 - 4 \left(\frac{1}{d}\right) (d+1) + \left(\frac{n}{d^2}\right) (d+1)^2 \sum_i P(E_i|\rho)^2 \right]. \quad (4)$$

Equating the two implies that

$$\sum_i P(E_i|\rho)^3 = \left(\frac{d^2}{n^2}\right) \left(\frac{1}{d(d+1)^2(d+2)}\right) \times \left[n(5d^2 + 14d + 9) \sum_i P(E_i|\rho)^2 - 4d^2 - 12d \right], \quad (5)$$

which indeed relates

$$\sum_i P(E_i|\rho)^2 = \left(\frac{d}{n}\right) \frac{2}{d+1} \quad (6)$$

$$\sum_i P(E_i|\rho)^3 = \left(\frac{d}{n}\right)^2 \frac{6}{(d+1)(d+2)}, \quad (7)$$

which I derive in the paper, and is equivalent to $\text{tr}(\rho^2) = \text{tr}(\rho^3) = 1$. It turns out that this is enough to pick out pure ρ .