From a lower-bound on the variance to the Jordan product?

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I'm a researcher in quantum foundations, and over the last few months, I've become fascinated by the representation of quantum mechanics furnished by measurements which form a complex-projective 3-design. On the one hand, such measurements are informationally-complete, so that density matrices can be substituted for probability-assignments on their outcomes. On the other hand, not all probability-assignments correspond to valid density matrices. For a 3-design in particular, however, the equations which valid probability-assignments must satisfy are strikingly simple: a paper on this topic is forthcoming. In this series of blog posts, I'll be thinking aloud about how to go in the reverse direction, seizing on certain key features of the 3-design representation, and trying to re-derive quantum theory by appealing to principles close to the heart of QBism, a subjective Bayesian interpretation of quantum mechanics. The ancient Neoplatonists often described the soul's metaphysical journey as consisting of two alternating parts, a procession from the One into Many, and thereafter a reversion from the Many to the One. Having started from quantum mechanics as it is already given, I'll try to document my quest to find it again. What follows will be informal, but technical, meant for my fellow researchers: if you, a stranger, find some interest in these posts, I'm happy to answer any questions in the comments, or elsewhere. Without further ado...

We begin with the existence of an ideal measure-and-prepare reference device characterized by some conditional-probabilities $P(E_i|\sigma_j)$. *We'd like this to be unbiased. Should we assume an Urgleichung already?* In what follows, it is useful to recall that from the fundamental consistency-relation $P(E|\rho) = P(E|\sigma)\Phi P(E|\rho)$, we pick out a privileged subspace $S = \{x = P\Phi x\}$. As an inaugural assumption paying homage to nature's vitality, let us suppose that the second-moment (and so as well the variance) with respect to the reference device of any observable $x \in S$ satisfies a lower-bound if and only if $P(E|\rho)$ is a valid probability-assignment. *Need we assume $P(E|\rho) \in S$?* In particular, we suppose that, like the second-moment itself, this lower-bound is a linear function of $P(E|\rho)$ and a quadratic function of x. We must therefore must have for some three-index tensor A_{ijk}

$$\forall x \in S : \sum_i x_i^2 P(E_i|\rho) \geq \sum_{ijk} A_{ijk} x_i x_j P(E_k|\rho), \tag{1}$$

or better yet,

$$\sum_i x_i^2 P(E_i|\rho) - \sum_{ijk} A_{ijk} x_i x_j P(E_k|\rho) \geq 0 \tag{2}$$

$$\sum_{ij} x_i \left[\sum_k \Big(\delta_{ij}\delta_{ik} - A_{ijk} \Big) P(E_k|\rho) \right] x_j \geq 0 \tag{3}$$

$$\sum_{ij} x_i B[\rho]_{ij} x_j \geq 0. \tag{4}$$

We'd like $B[\rho]$ to be a symmetric matrix, and so we want A to be at least symmetric in the first two indices. *In fact, we probably want it to be fully symmetric.* We've thus constructed a matrix $B[\rho]$ for which $x^T B[\rho] x$ is non-negative on all $x \in S$. $P\Phi$ is the projector onto S, and thus $C[\rho] = (P\Phi)^T B[\rho] P\Phi \geq 0$, that is, $C[\rho]$ will be positive semi-definite if and only $P(E|\rho)$ is a valid probability-assignment. We might say that $C[\rho]$ is a local representation of ρ since its validity can be checked directly. *Note we will like to normalize $C[\rho]$.*

For comparison, in the quantum mechanical case, for an unbiased 3-design, I have derived

$$\forall x \in S : \langle X^2|\rho\rangle \geq \frac{d}{d+2}\Big[\langle X^2|\mu\rangle - 2\langle X|\mu\rangle\langle X|\rho\rangle\Big], \tag{5}$$

where μ is the state of complete ignorance, that is,

$$\sum_i x_i^2 P(E_i|\rho) \geq \frac{d}{d+2}\left[\frac{1}{n}\sum_i x_i^2 - 2\left(\frac{1}{n}\sum_i x_i\right)\left(\sum_i x_i P(E_i|\rho)\right)\right]. \tag{6}$$

We want a symmetric matrix out of it, so let

$$B[\rho]_{ij} = \sum_k \left[\delta_{ij}\delta_{ik} - \frac{d}{d+2}\left(\frac{1}{n}\right)\Big(\delta_{ij} - \delta_{ik} - \delta_{jk}\ \Big)\right] P(E_k|\rho) \tag{7}$$

$$B[\rho] = \mathrm{diag}\big[P(E|\rho)\big] - \frac{d}{d+2}\left(\frac{1}{n}\right)\Big[I - P(E|\rho)u^T - uP(E|\rho)^T\Big], \tag{8}$$

where u is the vector of all 1's. Noting that$P\Phi = (P\Phi)^T$, let

$$C[\rho] = \frac{1}{2}\left(\frac{n}{d}\right)\left(\frac{d+2}{d+1}\right) P\Phi B[\rho] P\Phi. \tag{9}$$

We've actually met this matrix before. Recall that "taking the Jordan product with ρ" acts on vectorized operators as $L[\rho] = \frac{1}{2}(\rho \otimes I + I \otimes \rho^T)$. Let

$$\mathcal{L}[\rho]_{ij} = \mathrm{tr}(E_i L[\rho](\sigma_j)) = \frac{d}{n}(\sigma_i|L[\rho]|\sigma_j) = \sum_{kl} \Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]\Phi_{kl} P(E_l|\rho). \tag{10}$$

We have $\rho \geq 0 \Leftrightarrow L[\rho] \Leftrightarrow \mathcal{L}[\rho] \geq 0$, and actually $\mathcal{L}[\rho] = C[\rho]$! *It would be good to rehearse the algebraic steps here, going in reverse.* We've thus recovered the

Jordan product almost out of thin air. Moreover, if we let $X = \sum_i x_i E_i$, the second-moment *with respect to a von Neumann measurement* is

$$\mathrm{tr}(X^2\rho) = \left(\frac{d}{n}\right) \sum_{ijkl} x_i x_j \Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]\Phi_{kl} P(E_l|\rho) \tag{11}$$

$$= \left(\frac{d}{n}\right) x^T \mathcal{L}[\rho] x, \tag{12}$$

from which the lower-bound on the second-moment *with respect to the reference device* given above can be derived, using the expression peculiar to a 3-design which allows $\Re[\mathrm{tr}(E_i\sigma_j\sigma_k)]$ to be expressed in terms of $P(E_i|\sigma_j)$, as well as the 2-design property. Notice that the simple act of moving a term from the RHS to the LHS and then projecting into S takes us from the lower bound on $\langle X^2|\rho\rangle_E$ to the exact value of $\langle X^2|\rho\rangle_{\mathrm{vn}}$.

Can we show that pure states must be idempotents? Notice if we require for a pure probablity-assignment $C[\rho]\Phi P(E|\rho) = P(E|\rho)$, this at least fixes the normalization. *We recall that in the quantum case, the eigendecomposition of $C[\rho]$ for a pure state is particularly simple.* It would be good to prove that $C[\rho]\Phi$ stabilizes $P(E|\rho)$ if and only if $P(E|\rho)$ is an extreme point, i.e. such that it can't be written as a sum of probability vectors in S which satisfy lower-bound on the variance. What does being an extreme point imply about the variance lower-bound itself? Does it mean there exists observables which saturate it?

We also want purity to mean perfect distinguishability, that is, the state ought to imply certainty for some measurement. *Does self-duality mean that an extreme point must be perfectly distinguishable?* If we assume self-duality, we want $P(\rho|S)\Phi P(E|\rho) = \gamma P(E|\rho)^T \Phi P(E|\rho) = 1$ for some special fixed constant γ which can be interpreted in terms of Bayes' rule. This implies a quadratic equation that $P(E|\rho)$ must satisfy. But moreover,

$$\gamma P(E|\rho)^T \Phi C[\rho] \Phi P(E|\rho) = 1, \tag{13}$$

which is a cubic equation in $P(E|\rho)$. We could also get scalar equations from

$$\forall n : u^T \Big(C[\rho]\Phi\Big)^n P(E|\rho) = 1. \tag{14}$$

At what point, and with what structure, can we derive a conjunction of p-norm spheres? Finally, suppose we want

$$u^T C[\rho]\Phi P(E|\tau) \propto P(\rho|\tau)? \tag{15}$$

What restrictions does this place on A_{ijk}?