Quantum Mechanics: An Introduction

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This will be the first of a series of posts building up quantum mechanics from its basic principles.

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Introduction

Quantum mechanics is a theory of physics that cannot really be derived from classical mechanics[1]. But there are a number of analogies between the two which make it easier to understand quantum mechanics.

The principles of quantum mechanics outlined below may seem difficult to understand. In fact, they are very easy to understand - it's just that they are hard to believe. There is no real explanation for why these axioms are what they are, and they represent perhaps the deepest mystery in all of science.

There are a number of analogies that can be drawn between classical and quantum mechanics. I will begin by explaining some classical concepts, and then discussing their quantum analogues. The idea of a "state" plays a central role in quantum mechanics, so I begin by explaining its classical analogue.

State space

Consider a particle constrained to move along a straight line. In classical mechanics, we have what is called the **state space** of this physical system; the state space is set of all possible positions and velocities (x, v) of the particle. The position could be any real number, and the velocity could be any real number, so the state space is essentially $\mathbb{R}^2$. A particular choice of values x and v determines the **state** $s = (x, v)$ of the particle; thus the state of the particle corresponds to a particular point in state space.

Here are some other examples

- For a particle constrained to move on a circle, the state space would be an infinitely long cylinder. The position of the particle is restricted to be some number $\theta \in [0, 2\pi]$ which tells you how far around the cylinder you are, and the velocity could be any real number (which tells you how far up or down you are on the cylinder).
- A system consisting of two particles moving on a straight line would have $\mathbb{R}^4$ as its state space (2 positions and 2 veclocities). A possible state for this system would be of the form (x_1, v_1, x_2, v_2).
- A system of N particles moving around in in 3D space would have $\mathbb{R}^{6N}$ as its state space. If they were just moving around in 1D space, the state space would be $\mathbb{R}^{2N}$.
- Different physical systems typically have different state spaces.

In quantum mechanics, every physical system also has a state space. But, as we will see, the state space is always a vector space V. Compare this with classical mechanics, where the state space could be for example a cylinder as above, which is not really a vector space.

In classical mechanics, the state s has a simple interpretation as giving the positions and velocities of the particles. The quantum mechanical state of a particle $\psi \in V$ encodes the *probabilities* of observing different values for position, velocity, energy, etc.

Observables

In principle, in classical mechanics all the information you could possibly know about the system at a particular time is contained in its state.

For example, for a system of N particles in $\mathbb{R}^1$, if you want to know the kinetic energy of some particle in the system, you look at the state $s = (x_1, v_1, \ldots, x_n, v_n)$ of the system, check the component v_i corresponding to the velocity value for that particle, and compute $mv_i^2/2$.

If for whatever reason you had some experiment where you needed to compute the product of, let’s say, the position of the first particle and the velocity of the second particle (maybe you have a machine that measures this number), you would again look at the state of the system, check both the x_1 and v_2 values, and multiply them.

Both of these things, $mv_i^2/2$ and x_1v_2, are functions on $\mathbb{R}^{2N}$; they are functions on the state space. Any observable property of the system at a particular time, an **observable**, corresponds to a function on the state space.

In quantum mechanics, here is how observables are handled: each observable corresponds to a particular choice of basis for the state space V. Each basis vector represents a possible numerical value for the observable.

For example, for a particle on a line, the momentum of the particle can be observed; momentum is an observable. Thus there is a basis for momentum. Each momentum basis vector corresponds to some value for the momentum, ie. there may be one vector corresponding to $6\frac{m}{s}$, another for $-7\frac{m}{s}$, etc. The position of the particle can also be observed, so it is also an observable, and for position we can repeat everything I just said about momentum.

Recall that a basis for a vector space of dimension n consists of n vectors; thus in cases where observables can take on infinitely may values, we must be dealing with an infinite dimensional vector space[2]. In the example above with momentum, the state space is probably going to be infinite dimensional, since momentum can typically take on infinitely many values. On the other hand, there are systems where the state space is finite dimensional, as we will see, and thus where observable properties can only take on finitely many values.

Measurement

Perhaps the most unusual feature of quantum mechanics is measurement.

In classical mechanics, measurement of an observable is a straightforward process, so straightforward that there is really nothing much to say about the it. The values of all observables at a particular time are determined in an obvious way by the state. For example, if the state of a particle on a line is (x, v), measurement of its velocity would yield v, measurement of it's kinetic energy would yield $mv^2/2$, measurement of its position would yield x, and so on; and that's the end of the story.

The basic idea of measurement in quantum mechanics is this. In the last section I said that each observable has its own basis for the state space, and each such basis vector represents a possible numerical value for the observable. Let's say there is a basis vector for momentum, call it $\boldsymbol{v}_4$, representing momentum $4\frac{m}{s}$. If the system is in the state $\boldsymbol{v}_4$ and the momentum measured, the momentum will be found to be $4\frac{m}{s}$. If it is in the state $\boldsymbol{v}_8$ representing momentum $8\frac{m}{s}$, the momentum will be found to be $8\frac{m}{s}$. And so on. But what if the state of the system is not in a momentum basis state? This is where we begin to find remarkable departures from classical physics.

Since the state space is a vector space V, we can take linear combinations of states, like

$$2\boldsymbol{v}_4 + 7\boldsymbol{v}_8$$

Any such linear combination represents another state. If the momentum is measured when the system is in a state like this, where the coefficients of both $\boldsymbol{v}_4$ and $\boldsymbol{v}_8$ are non-zero, the following are all true

- The momentum could be found to be 4
- The momentum could be found to be 8
- The momentum cannot be found to be anything other than 4 or 8
- It is impossible, even in principle, to predict with certainty whether the momentum will be 4 or 8
- That is, the measurement has an inherent randomness
- The larger the coefficient of $\boldsymbol{v}_4$, the more likely the momentum will be found to be 4; and similarly for $\boldsymbol{v}_8$
- This lack of certainty in the result has nothing to do with the precision of the measuring equipment or anything like that; the randomness is inherent to quantum mechanics
- Prior to measurement, the momentum was **indeterminate**; it was not 4, and it was not 8; it did not have a specific value[3]

In summary, in the state $2\boldsymbol{v}_4 + 7\boldsymbol{v}_8$, the momentum is indeterminate[4]. The system does not have a definite momentum, and it is said to be in a superposition[5] of momentum states. Informally, we could say it's in a combination of the two states. Consider how radically different this is from classical mechanics, where a particle always has a well defined momentum.

Everything I've just said about momentum applies to all observables.

The last feature of measurement in quantum mechanics is this

- After measurement, if the momentum is found to be 4, the system is now in state $\boldsymbol{v}_4$ (and similarly if the momentum is found to be 8)
- That is, the process of measurement has pushed the state of the system into one of the momentum basis states

This change of the state of the system upon measurement is called **wave function collapse**. The term "wave function" refers to the state of the particle, which "collapses" to one of the basis states upon measurement. Again, we have another dramatic departure from classical mechanics; the process of measurement itself changes the state of the system. This is again inherent to the physics of quantum mechanics, and has nothing to do with the crudeness of the measuring equipment; it applies no matter how carefully and delicately the measurement is made.[6]

There is one class of "exceptions" to all of this; in this example, if the state was $\boldsymbol{v}_4 = 1\boldsymbol{v}_4 + 0\boldsymbol{v}_8$ to begin with, then the probability of getting momentum 8 upon measurement is 0%, the probability of getting 4 is 100%, and after measurement the state must still be $\boldsymbol{v}_4$. (And similarly if the state was $\boldsymbol{v}_8$ to begin with.) These are not really exceptions of course, since everything in this paragraph still conforms to the rules of measurement.

We can see now how the vector space structure of the state space contributes to the physics of quantum mechanics. A state $\boldsymbol{\psi}$ can be built up by taking a sum or linear combination of some other states:

$$\boldsymbol{\psi} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$$

Informally, the state $\boldsymbol{\psi}$ is a little bit like $\boldsymbol{v}_1$, and a little bit like $\boldsymbol{v}_2$; measuring the system when it is in state $\boldsymbol{\psi}$ will be like measuring it when it is in state $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$. The larger a particular coefficient is, the more $\boldsymbol{\psi}$ is like the corresponding state.

Potentials/Energy and Time Evolution

In classical mechanics, there is typically some equation that tells you how things change in time. In Newtonian mechanics this is $\vec{F} = m\vec{a}$; in more advanced classical mechanics this is Lagrange’s or Hamilton’s equations. Often this is where the concept of energy comes into play.

Returning to the state space for a pair of particles on a straight line, suppose the particles are moving around subject to some potential/force. As time passes, their positions and velocities change, and so the state s of the system changes; at each time t, the system is represented by a different point in state space $s(t)$.

In other words, as time passes, the state of the system traces out a curve in state space. The changing of the state of the system with time is called **time evolution** and is determined in some way by the potential energies of the particles.

For example, consider again a single particle on a straight line (recall that the state space is $\mathbb{R}^2$; the first component is position, and the second velocity), experiencing no force (no potentials, etc.), moving at constant velocity v_0; the time evolution is given by a curve $s(t)$ in $\mathbb{R}^2$

$$s(t) = (v_0 t + x_0, v_0)$$

In quantum mechanics, as in classical mechanics, the state of the particle typically changes with time and traces out a curve $\boldsymbol{\psi}(t)$ in state space. The time evolution is determined by the **Schrodinger equation**, which involves the total energy of the system.

Determinism

Classical mechanics is a deterministic theory. If you have 100% perfect and complete information about the system now, you automatically have 100% perfect and complete information about the system at all past and future times. That is, if the state of the system $s(t_0)$ is known at a particular time t_0, then $s(t)$ is known for all other times t.

Everything in the preceding paragraph is true in quantum mechanics for the quantum mechanical state $\boldsymbol{\psi}(t)$, between measurements - time evolution in quantum mechanics is a completely deterministic process. If you know $\boldsymbol{\psi}(t)$ at time $t = t_0$, you know $\boldsymbol{\psi}(t)$ for all other times. It is measurement that breaks the determinism of time evolution and introduces randomness into the system.

Studying a physical system using quantum mechanics

The two crucial ingredients needed for studying a physical system using quantum mechanics are the state space and the various bases of different observable properties.

The "bare" state space V, a vector space, has no information, it is meaningless. It obtains meaning when we figure out what the vectors represent. When we can say, "Ok, $\begin{pmatrix}1\\0\end{pmatrix}$ represents an electron with this much angular momentum about the z-axis," then we can start doing physics.

Figuring out what the state space is for a physical system, and what the basis is for a particular observable property, is not an entirely straightforward task[7] as it it typically is in classical mechanics. It is based on a combination of theoretical grounds, experimental grounds, and common sense. I will do this on a case by case basis for a few different physical systems[8]. But first, I will explain in more detail the 4 axioms above, and develop some calculational tools.

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1. If you’re not sure what terms like "classical" and "classical mechanics" mean, you can think of them as referring to the Newtonian mechanics you learned in high school.↩
2. I will explain the mathematical details of infinite dimensional vector spaces later↩
3. Of course in the distant past, maybe the momentum was 4, or maybe it was 8; but for as long as the state has been $2\boldsymbol{v}_4 + 7\boldsymbol{v}_8$, the momentum has not been 4 or 8.↩
4. I will used "well defined" and "definite" as antonyms for indeterminate.↩
5. "superposition" is a physics synonym for "linear combination"↩
6. You may argue that even in classical mechanics, we cannot in practice measure a system without changing it, by bouncing particles off of it and exchanging energy/momentum, for example. The point is that in classical mechanics an effect of measurement on the system is not built into the theory; the effect can be arbitrarily small. In quantum mechanics there is a substantial, built-in effect of measurement; in the above example the final state, either $\boldsymbol{v}_4$ or $\boldsymbol{v}_8$, is substantially different from the starting state $2\boldsymbol{v}_4 + 7\boldsymbol{v}_8$, as we will see.↩
7. Of course, by this point in time the details have been worked out for a wide variety of quantum mechanical systems.↩
8. To some extent I have already done this a little; I deduced above that the state space for a particle on a line must be infinite dimensional.↩