Motivating bialgebras and Hopf algebras

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This is part of an ongoing series on quantum groups and knot invariants, the first few entries of which were posted on Tumblr last year. This particular post doesn't require any familiarity with those earlier installments.

The goal of this post is to introduce (and provide some motivation for why we should care about) a class of algebraic structures called **Hopf algebras**. We assume familiarity with the basic concepts of groups, rings, vector spaces and fields, but not much beyond that. All rings are assumed to be unital (and all ring homomorphisms preserve the identity), but we do not require the rings to be commutative.

Algebras over a field

We begin by defining (associative) **algebras**.

Morally speaking, an algebra over a field $\mathbb{K}$ (algebras can be defined more generally over any commutative ring but we will not need this more general definition for this series) is a vector space that also has the structure of a ring, in such a way that these two structures are "as compatible as possible". A vector space has notions of "addition" and "scalar multiplication", while a ring has its own notions of "addition" and "multiplication". For an algebra we want those two notions of addition to be the same, and the two notions of multiplication to be as similar as they can be given that, for a general ring, multiplication is not necessarily commutative.

Definition 1

An *algebra* is a $\mathbb{K}$-vector space A together with two linear maps,

$$\mu : A \otimes A \to A$$

and

$$\eta : \mathbb{K} \to A$$

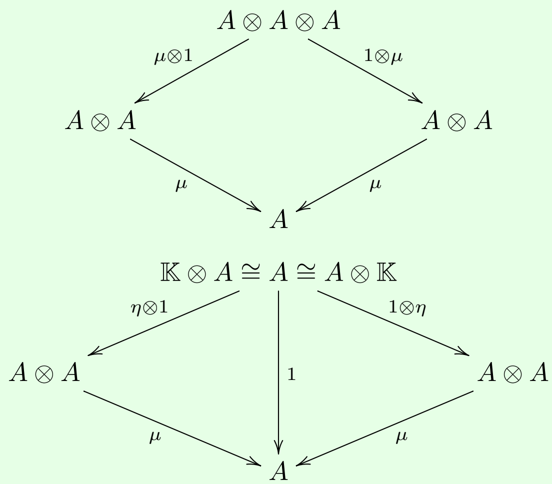
such that

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c) \qquad \forall\, a, b, c \in A \tag{1}$$

$$\mu(a, \eta(1_{\mathbb{K}})) = a \qquad \forall\, a \in A \tag{2}$$

$$\mu(\eta(1_{\mathbb{K}}), a) = a \qquad \forall\, a \in A\,. \tag{3}$$

Equivalently, we can express this by requiring that the following two diagrams commute:



Even more compactly, we can remember the slogan: an associative algebra is a monoid in the category of vector spaces.

Here we began with a vector space A and then specified some additional linear maps between vector spaces that (we claim) gave our vector space A a suitable ring-structure. There is also an equivalent definition going the other way: we begin with a ring and impose a condition in terms of morphisms between rings that give this ring the structure of a vector space.

Before giving this alternative definition and proving it is equivalent, we had better check that the algebra have defined really does have the structure of a ring.

Lemma 2

Let A be an algebra over a field $\mathbb{K}$. Then A is a ring.

Proof We define multiplication of elements $x, y \in A$ by $xy = \mu(x, y)$. We define the identity element in A by $1_A = \eta(1)$ where 1 is the identity element of our base field $\mathbb{K}$.

To show that A is a ring, we need to show that is an abelian group under addition (something we get for free from the fact that A is a vector space), that it is a monoid under multiplication (which we get from the three equations we gave in Definition 1) and that multiplication distributes over addition (which we get from the fact we specified that μ was a linear map). There is nothing else to prove.

Now we have:

Theorem 3

Let A be a ring. Then A is an algebra over a field $\mathbb{K}$ if and only if there is a ring monomorphism $\phi\colon\mathbb{K}\to Z(A)$, where $Z(A)$ denotes the center of A.

Proof We start with the "only if" direction. Suppose that A is an algebra over $\mathbb{K}$.

By Lemma 2 we know that A is a ring. We claim that the required morphism from $\mathbb{K}$ to the center of A is precisely the linear map η. By definition we have $\eta(1)=1_A$ so this map preserves multiplicative identities. Moreoever, since η is a $\mathbb{K}$-linear map whose domain is $\mathbb{K}$ itself it is in fact a ring homomorphism: since $\eta(\lambda)=\lambda\eta(1)=\lambda 1_A$ for any $\lambda\in\mathbb{K}$ we must have $\eta(\lambda_1\lambda_2)=\eta(\lambda_1)\eta(\lambda_2)$.

Since for any $\lambda_1,\lambda_2\in\mathbb{K}$ we have $\lambda_1\lambda_2=\lambda_2\lambda_1$ we must also have

$$\eta(\lambda_1)\eta(\lambda_2)=\lambda_1\lambda_2 1_A=\lambda_2\lambda_1 1_A=\eta(\lambda_2)\eta(\lambda_1)\ .$$

It follows that $\eta(A)\subseteq Z(A)$.

For the "if" direction, suppose A is a ring and that $\rho\colon\mathbb{K}\to A$ is a ring homomorphism with $\rho(\mathbb{K})\subseteq Z(A)$. To show that A is a vector space over $\mathbb{K}$ we have to show that it is an abelian group under addition (which it must be because it is a ring) and define a suitable notion of scalar multiplication.

We set $\lambda\cdot a:=\rho(\lambda)a$. This operation is commutative, as required, precisely because $\rho(\lambda)$ is ine center of A. It also has the required distributivty properties because ρ is a ring morphism. This completes the proof.

As well as the definition of associative algebras as particularly nice rings, or as a "monoids in the category of vector spaces", a third perspective is possible. This is a somewhat more concrete view which can be useful for doing calculations. We think of algebras as not merely abstract vector spaces but as vector spaces with some fixed basis, and define multiplication directly in terms of the result of multiplying two basis elements together. We are free to choose the basis however we want, so we can fix one of the basis elements to be the multiplicative identity element $\eta(1)=1_A$.

Definition 4

Let A be an algebra over $\mathbb{K}$ with basis $\{e_i\}$ such that $e_1=1_A$. Then we define the *structure constants* $c_{i,j}^k\in\mathbb{K}$ by

$$e_ie_j=\sum_k c_{i,j}^k e_k.$$

In this notation, the associativity axiom and the fact that e_1 should act as a unit with respect to multiplication can both be captured as a relationship between structure constants.

Proposition 5

Let A be an algebra over $\mathbb{K}$ with basis $\{e_i\}$ such that $e_1=1_A$ and structure constants $\{c^k_{i,j}\}$. Then

$$\sum_m c^m_{i,j} c^n_{m,k} = \sum_k c^m_{j,k} c^n_{i,m}$$

and

$$c^n_{1,j} = c^n_{j,1} \ .$$

Proof Since the algebra is associative we must have

$$(e_i e_j)\, e_k = e_i\, (e_j e_k) \ .$$

Expanding both sides of this equation in terms of the structure constants we eventually obtain

$$\sum_{m,n} c^m_{i,j} c^n_{m,k} e_n = \sum_{m,n} c^m_{j,k} c^n_{i,m} e_n$$

from which the claimed identity follows at once.

For the unit, we have that $\eta(1)e_j = e_1 e_j = e_j e_1 = e_j \eta(1)$. Expanding both sides ofthis equation in terms of the structure constants gives

$$\sum_n c^n_{1,j} e_n = \sum_n c^n_{j,1} e_n$$

and again the claimed identity follows.

Before giving a few examples of algebras (see below), we note one way of constructing new algebras from existing algebras. First we fix some notation: for any two vector spaces U and V let $\tau : U \otimes V \to V \otimes U$ be the unique linear map which maps each basis element $u \otimes v$ to $v \otimes u$. Now:

Proposition 6

Let (A, μ_A, η_A) and (B, μ_B, η_B) be two algebras over a field $\mathbb{K}$. Then the tensor product $A \otimes B$ has a natural algebra structure given by

$$\mu_{A\otimes B} := (\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B)$$
$$\eta_{A\otimes B} := \eta_A \otimes \eta_B \ .$$

Proof The maps $\mu_{A\otimes B}$ and $\eta_{A\otimes B}$ are clearly linear (they are the composition or tensor product of linear maps). That they satisfy the required axioms can be checked by a direct calculation, which we will omit here.

We end this section by defining an algebra homomorphism: a linear map between two algebras that preserves the algebra stucture. In the diagram notion we have been using, this is:

Definition 7

Given two algebras A and B, an algebra homomorphism is a linear map $\phi\colon A \to B$ such that the diagrams below commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ {\scriptstyle \phi\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ B \otimes B & \xrightarrow[\mu_B]{} & B \end{array} \qquad \begin{array}{ccc} \mathbb{K} & & \\ {\scriptstyle \eta_A}\downarrow & \searrow^{\eta_B} & \\ A & \xrightarrow[\phi]{} & B \end{array}$$

Some examples You are probably already familiar with many examples of associative algebras. You might not be familiar with all of these, but we will return to many of them later.

- The prototypical example is the algebra of polynomials in one variable over $\mathbb{K}$, denoted $\mathbb{K}[x]$. The unit is the constant polynomial 1 and multliplication is given in the natural way. More generally the ring of polynomials in multiple (commuting) variables $\mathbb{K}[x_1, x_2, ...x_n]$ is also an algebra, as is the ring of polynomials in multiple non-commuting variables $\mathbb{K}\langle x_1, x_2, \dots x_n\rangle$. These algebras all are infinite dimensional (as vector spaces).
- The complex numbers $\mathbb{C}$ are a two dimensional algebra over the real numbers $\mathbb{R}$. More generally any field extension is an algebra over its base field. And trivially any field $\mathbb{K}$ is an algebra over itself.
- If G is a (finite) group, then let $\mathbb{K}G$ be the vector space whose basis is the elements of G. This vector space becomes an algebra if we define the product of two basis elements to be their product in G, and extend this $\mathbb{K}$-linearly to all elements.
- If G is a group, then let $\mathbb{K}[G]$ be the vector space whose basis is the set of all linear maps from $G \to \mathbb{K}$.
- The Hecke algebra and Temperley-Lieb algebras defined in an earlier post in this series are associative algebras in this sense. Recall in particular that the Temperley-Lieb algebra is the algebra with generators $U_1, U_2, \dots, U_n$ subject to the relations

$$\begin{aligned} U_i^2 &= \delta U_i && \text{for all } i \\ U_iU_jU_i &= U_jU_iU_j && \text{if } |i-j| = 1 \\ U_iU_j &= U_jU_i && \text{if } |i-j| > 1 \end{aligned}$$

  where $\delta \in \mathbb{K}$.
- For any positive integer n, the ring $M_{n,n}(\mathbb{K})$ of $\mathbb{K}$-valued matrices is an algebra. This example shows in particular that while the image of $\mathbb{K}$ under η must be contained in the center of an algebra, it need not be equal to it.
- Many other matrix algebras are of interest. We will be particularly interested in $U(\mathfrak{g})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. In

particular, $U(\mathfrak{sl}_2)$ is the (infinite dimensional!) algebra with generators e, f, h and subject to the relations

$$
\begin{aligned}
he - eh &= 2e \\
hf - fh &= -2f \\
ef - fe &= h
\end{aligned}
$$

- If Q is a quiver (i.e. a directed multigraph) then the path algebra $\mathbb{K}Q$ is defined to be the vector space with basis all (directed) paths in Q (including the paths of zero length: the vertices). Multiplication of two paths is given by concatenation (if the start and ends of the two paths agree) or is defined to be zero (if they do not).
- If V is any vector space over $\mathbb{K}$ then let

  $$T^k V = V^{\otimes k} = \overbrace{V \otimes V \otimes \cdots \otimes V}^{k \text{ times}} ,$$

  where in particular $T^0 V = \mathbb{K}$, and let

  $$T(V) := \bigoplus_{k \in \mathbb{N}_0} T^k V .$$

  We give this new (infinite-dimensional) vector space the structure of an associative algebra – the *tensor algebra* – by using the natural isomoprhism $T^i V \otimes T^k V \to T^{i+j} V$ and extending it linearly to all of $T(V)$. That is, given $u \in T^i V$ and $w \in T^j V$ where

  $$
  \begin{aligned}
  u &= u_1 \otimes u_2 \otimes \cdots u_i \quad \text{and} \\
  w &= w_1 \otimes w_2 \otimes \cdots w_j
  \end{aligned}
  $$

  we define their product to be

  $$uv = u_1 \otimes u_2 \otimes \cdots \otimes u_i \otimes w_1 \otimes \cdots \otimes w_j .$$

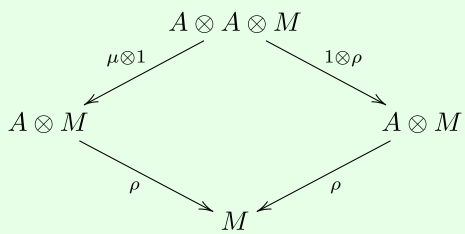
Representations of groups and algebras

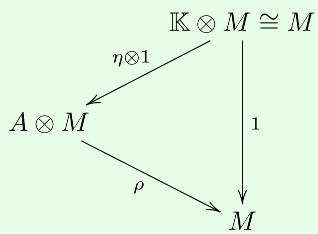
Let G be a (not necessarily finite) group. By a representation of G we mean a map $\rho : G \to \mathrm{GL}(V)$. the set of invertible linear transformations acting on a (finte dimensional) vector space V. Because linear transformations on a finite dimensional vecotr space are equivalent to $n \times n$ matrices, the map ρ extends to a map from $\mathbb{K}G \to \mathrm{End}(V)$.

It is slightly more fashionable (useful?) to talk about modules rather than representations. A module is to an algebra (or any ring) as a vector space is to a field. More formally

Definition 8

Let $\mathbb{K}$ be a field and A an algbera over that $\mathbb{K}$. A (left) A-module M is a vector space over $\mathbb{K}$ together with a map $\rho \colon A \otimes M \to M$ such that the following diagrams commute:





There is an equivalent notion of right module, defined in the obvious way. Note that any algebra A has two 'trivial' modules: the zero module (on which every element of A acts like zero) and the algebra A itself (here ρ is called the regular representation of A).

To help motivate the importance of modules, we recall some ring theory.

Definition 9

If R is a ring and $I \subseteq R$ is a subset of R closed under addition, then I is called:

- A left ideal if $RI = I$.
- A right ideal if $IR = I$.
- A two-sided ideal (or simply an ideal) if it is both a left and a right ideal.

The singleton set $\{0\}$ and the full ring R are both ideals of R. We can order the set of ideals of R by inclusion to give them the structure of a lattice, where $\{0\}$ is the common least element and R is the common greatest element. If $I \neq R$ is an ideal of R we say that I is **maximal** if, for any other ideal J, $I \subseteq J$ implies that either $J = I$ or $J = R$.

An ideal in a ring is roughly analogous to the idea of a normal subgroup in a group. One difference is that, in general, an ideal is not itself a ring. However, given a ring R and a two-sided ideal $I \subseteq R$ we can construct a new ring, called the **quotient ring**, in much the same way we use a normal subgroup to construct a quotient group.

Definition 10

Suppose R is a ring and $I \subseteq R$ is a two-sided ideal. Define an equivalence relation $\sim$ on R by $x \sim y$ if $(x - y) \in I$. Then the *quotient ring* R/I is the set of equivalence classes under $\sim$, with addition and multiplication inherited from R. That is:

$$[x] + [y] = [x + y]$$
$$[x][y] = [xy] \ .$$

Note that there is an implicit claim here that addition and multiplication are well-defined (i.e. that they are independent of the choice of representative of each equivalence class). Once this claim is checked it is straightforward to see that R/I must be a ring.

If you know a little group theory, the following result should not surprise you:

Theorem 11

Let R and S be rings and $\phi : R \to S$ a ring homomorphism. Then

- $\ker \phi \subseteq R$ is a two-sided ideal
- $R/\ker \phi \cong \operatorname{im} \phi \subseteq S$

It is clear from the definition that (left) ideals of an algebra A are (left) A-modules. Further, the quotient ring A/I is also an A-module: $\rho(a \otimes [x]) = [ax]$. This then is one motivation to study modules of an algebra.

Motivation #1

The concept of 'module' generalizes the concept of 'ideals' and 'quotient rings'.

A morphism between modules is a linear map that respects the action of the algebra on these modules. In terms of diagrams:

Definition 12

Let M and N be modules for a $\mathbb{K}$-algebra A. A module morphism is a linear map $f : M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes M & \xrightarrow{1 \otimes f} & A \otimes N \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

A invertible module morphism $f : M \to N$ whose inverse $f^{-1} : N \to M$ is also a module morphism is called an isomorphism. Two modules are said to be isomorphic if there exists an isomorphism between them. The category of A-

modules is the category whose objects are the modules of A and whose arrows are the module morphisms between A.

Isomorphic modules are "essentially the same". The associated representations are not necessarily identical but are equivalent as matrices: we have $\rho_M(a) = f^{-1}\rho_N(a)f$ for every $a \in A$.

If I and J are ideals with $I \subsetneq J$ then, as A-modules, there is obviously a map $I \to J$. We say in this case that I is a submodule of J. Indeed, recall that the notion of category generalizes the notion of partially ordered set. The lattice of ideals of A is therefore contained within the category of A-modules: morphisms between modules generalize the notion of inclusions between ideals.

Motivation #2

> The category of A-modules preserves the lattice structure of ideals.

Clearly, just as the zero ideal is contained within every ideal, every submodule contains the trivial zero module as a submodule. The modules with no submodules except for themselves and this trivial module are called **simple**. Simple modules turn out to be very important to the theory of modules (as usual, a reader familar with enough group theory to have heard of the Jordan-H/"{o}lder theorem should not be surprised by this).

It is not true that every simple module can be identified with an ideal. However, something close to this is true:

Motivation #3

> If S is a simple A-module then there exists some maximal ideal $J \subseteq A$ such that $S \cong M/J$.

Proof We only sketch a proof here. The key is a version of the ismorphism theorem for modules which we state but will not prove: if $\phi : M \to N$ is a module morphism then $\ker\phi$ is a submodule of M and $\operatorname{im}\phi$ is a submodule of N.

In particular, A is itself an A-module (left or right) and any (left) ideal is a (left) module. If $I \subseteq A$ is an ideal of A such that A/I is simple, then I must be maximal: otherwise there exists an ideal $J \subseteq A$ with $I \subsetneq I$ and A/J is a submodule of A.

Now suppose S is a (non-zero) simple (left) module and choose some $s \in S$. Define a module morphism $\phi : A \to S$ by $\phi(a) = as$. This map is clearly not identically zero, so – as $\phi(A)$ must be a submodule of S – it can only be equal to S itself. S is then isomorphic to $A/\ker\phi$ and therefore $\ker\phi$ must be a maximal ideal.

So ideals are not merely an arbitrary special case of modules: the connection between modules and ideals goes in both directions.

Having motivated the idea that modules are important for understanding algebras, we turn briefly to the question of constructing modules.

Given any algebra A, we get an A-module for free: A itself is a module. This is obvious if we compare the diagrams above to the diagrams that appeared after Definition 1. In fact, we get infinitely many modules: the direct sum of any two modules is again a module, so $A \oplus A$, $A \oplus A \oplus A$, ... and so on are all A-modules. Modules of this form are called free modules.

For groups, the situation is even better. If M is a module for a group algebra $\mathbb{K}G$, then so is $M \otimes M$. The action of G on M is given by

$$g \cdot (m \otimes n) = (g \cdot m) \otimes (g \otimes n)$$

for all $g \in G$. We also have a "trivial" one-dimensional module, the underlying field $\mathbb{K}$. Here the action of $\mathbb{K}G$ on $\mathbb{K}$ is given by $g \cdot 1 = 1$ for all $g \in G$. Furthermore, if $\rho : G \to \mathrm{GL}(V)$ is a representation, then there is a second representation – called the **dual representation** or sometimes the contragredient representation – denoted ρ^* and defined by $\rho^*(g) = \rho(g^{-1})^H$, where M^H denotes the **conjugate transpose** of the matrix M.

For more general algebras, none of these constructions necessarily works. Indeed most algebras do not have any notion of 'inverses' of their basis elements.

However, if there is a map $f : A \to B$ and V is a B-module, then we can always extend V to an A-module by defining an action $a \cdot v = f(a) \cdot v$. In particular, if B is a quotient algebra of A then every B-representation can be extended in this way to an A-representation. On the other hand, if B is merely a subalgebra of A then there is no obvious way to turn a representation of B into a representation of A (though we can, of course, turn every representation of the larger algebra A into a represenation of the smaller algebra B).

Some examples

- Let $\mathbb{T}_4(\mathbb{K})$ be the algebra of upper-triangular 4×4 matrices and let $I \subset \mathbb{T}_4(\mathbb{K})$ be the (left) ideal consisting of matrices of the form

$$\begin{pmatrix} 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} .$$

  Then I is a maximal ideal, and $\mathbb{T}_4(\mathbb{K})/I$ is (isomorphic to) the simple one-dimensional module on which $M = (m_{i,j})$ acts like multiplication by the leading diagonal element $m_{1,1}$.

- Let A_n be the Temperley-Lieb algebra on $n-1$ generators $U_1, U_2, \ldots, U_{n-1}$, and let I be the (two-sided) ideal of A_n with basis every non-trivial word in these $n-1$ generators. Then I is a maximal ideal and A_n/I is (isomorphic to) the simple-one dimensional module on which each generator U_i acts like zero.

- If C_n is the cyclic group $\langle g | g^n = 1 \rangle$ then there is a (non-trivial) one-dimensional representation $\rho : C_n \to \mathbb{K}$ given by $\rho(g) = \exp\left(\frac{2\pi i}{n}\right)$. The dual of this representation is given by $\rho^*(g) = \exp\left(-\frac{2\pi i}{n}\right)$
- If C_3 is the cyclic group $\langle g | g^3 = 1 \rangle$ then the regular representation is map from C_3 to $M_3(\mathbb{K})$ given by

$$\rho(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

  The extenstion to the regular representation for C_n should be obvious.

- The universal enveloping algebra $U(\mathfrak{sl}_2)$ has a representation $\rho_1 : U(\mathfrak{sl}_2) \to M_2(\mathbb{K})$ given by

$$\begin{aligned} \rho_1(e) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \rho_1(f) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \rho_1(h) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{aligned}$$

- More generally, the same universal enveloping algebra $U(\mathfrak{sl}_2)$ has a representation $\rho_n : U(\mathfrak{sl}_2) \to M_{n+1}(\mathbb{K})$ given by

$$\begin{aligned} \rho_n(e) &= \begin{pmatrix} 0 & n & 0 & \dots & 0 & 0 \\ 0 & 0 & (n-1) & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \\ \rho_n(f) &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & n & 0 \end{pmatrix} \\ \rho_n(h) &= \begin{pmatrix} n & 0 & 0 & \dots & 0 & 0 \\ 0 & (n-2) & 0 & \dots & 0 & 0 \\ 0 & 0 & (n-4) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(n-2) & 0 \\ 0 & 0 & 0 & \dots & 0 & -n \end{pmatrix} . \end{aligned}$$

- The Temperley-Lieb algebra has a representation given by

$$U_i \mapsto I_{2^{i-1}} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I_{2^{n-i-2}} \ ,$$

  where $q + q^{-1} := \delta$. Since the Temperley-Lieb algebra is a quotient of the Hecke algebra, this map in turn defines a representation of the Hecke algebra.

Coalgebras

One advantage of defining algebras in terms of commutative diagrams in some category is that, as good category theorists, we can then ask what happens if we reverse the direction of all the arrows. As you might have guessed, we obtain the definition of a new algebraic structure called a **coalgebra**.

Definition 13

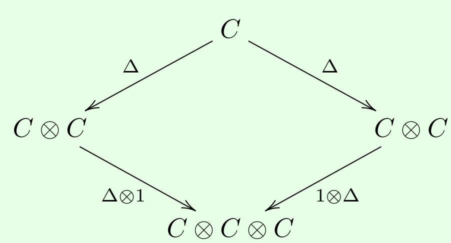
A *coalgebra* is a $\mathbb{K}$-vector space C together with two linear maps

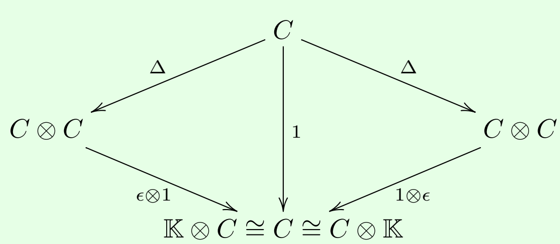
$$\Delta : C \to C \otimes C$$

and

$$\epsilon : C \to \mathbb{K}$$

such that the following two diagrams commute:





Coalgebras are not just a curiosity, but arise naturally in combinatorics. If we think about multiplication (in an algebra) as a way of combining two elements to obtain a third, then comultiplication – the linear map $\Delta : C \to C \otimes C$ – can be thought of as a way of decomposing an element into all the different possible pairs that could have been combined to obtain it. We will see this in the examples later.

It is worth unpacking the definition of Δ a little bit. If $x \in C$ then the fact Δ is a linear map means that

$$\Delta(x) = \sum_{i=1}^{n} a_i \otimes b_j ,$$

where the exact choice of a_i and b_j is not unique. We can fix a basis $\{v_i\}$ for C, and express $\Delta(x)$ as a sum in terms of this basis, but if haven't done this there are lots of possible different choices of $\{a_i\}$ and $\{b_i\}$.

When the exact choice of choices of $\{a_i\}$ and $\{b_i\}$ is not important, we can use **Sweedler notation**. We write the sum $\Delta(x) \in C \otimes C$ as

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \ .$$

Here $x_{(1)}$ and $x_{(2)}$ are not specific elements but merely placeholders representing an arbitrary summand of the whole sum. With this notation, the coassociativty of Δ – the fact that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ – can be described as

$$\sum \Delta(x_{(1)}) \otimes x_{(2)} = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = \sum x_1 \otimes \Delta(x_{(2)}) \ .$$

The relationship between Δ and ϵ – the fact that $(\epsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \epsilon) \circ \Delta$ – can be expressed as

$$x = \sum \epsilon(x_{(1)}) x_{(2)} = \sum \epsilon(x_{(2)}) x_{(1)} \ .$$

Given any vector space V recall that we can define the dual space $V^\star$ by

$$V^\star := \mathrm{Hom}(V, \mathbb{K}) \ .$$

In particular, an algebra A and a coalgebra C both have dual spaces. From the duality of their definitions, we might hope that the dual space of an algebra can be given a coalgebra structure in some natural way (and vice versa). This turns out to be almost, but not quite, true:

Proposition 14

The dual vector space of a coalgebra is an algebra.

Proof We define multiplication of two elements $f, g \in C^\star = \mathrm{Hom}(C, \mathbb{K})$ by

$$(fg)(x) = \sum f(x_{(1)}) g(x_{(2)})$$

for all $x \in C$, where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. This multiplication is associative precisely because Δ is coassociative. For the identity element, recall that

$\epsilon : C \to \mathbb{K}$ is a linear map and hence $\epsilon \in C^\star$. Now for any linear map $f \in C^\star$ we must have

$$\begin{aligned} f(x) &= f\left(\sum \epsilon(x_{(1)})x_{(2)}\right) \\ &= \sum \epsilon(x_{(1)})f(x_{(2)}) \\ &= (\epsilon f)x \end{aligned}$$

and similarly

$$\begin{aligned} f(x) &= f\left(\sum \epsilon(x_{(2)})x_{(1)}\right) \\ &= \sum f(x_{(1)})\epsilon(x_{(1)}) \\ &= (f\epsilon)x \ . \end{aligned}$$

Suppose V is a finite dimensional vector space, so that $V \cong V^\star$. If $\{e_i\}$ is a basis for V, then $V^\star$ has dual basis $\{f_i\}$, where the linear functions $f_i : V \to \mathbb{K}$ are defined by

$$f_i\left(e_j\right) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad .$$

Proposition 15

The dual vector space of a finite dimensional algebra is a coalgebra.

Proof Here the key result we need is that, for a finite dimensional vector space A, we have $(A^\star \otimes A^\star) \cong (A \otimes A)^\star$. This means that if $\{e_i\}$ is a basis for A and $\{f_i\}$ is a basis for $A^\star$ then $\{f_i \otimes f_j\}$ is a basis for $(A \otimes A)^\star$. We define maps $\Delta : A^\star \to (A \otimes A)^\star$ and $\epsilon : A^\star \to \mathbb{K}$ by, for $g \in A^\star$,

$$\begin{aligned} \Delta(g) &= \sum_{i,j} g(e_i e_j) f_i \otimes f_j \\ \epsilon(g) &= g\left(\eta(1)\right) \ . \end{aligned}$$

To show that Δ is coassociative requires a little calculation.

First we note that

$$\left(\Delta \otimes 1\right)\Delta(g) = \sum_{i,j,p,q} g(e_i e_j) f_i(e_p e_q) f_p \otimes f_q \otimes f_j$$

and that

$$\left(1 \otimes \Delta\right)\Delta(g) = \sum_{i,j,p,q} g(e_i e_j) f_j(e_p e_q) f_i \otimes f_p \otimes f_j \ .$$

We can rewrite the second of these equations as

$$(1 \otimes \Delta)\,\Delta(g) = \sum_{i,j,p,q} g(e_p e_i) f_i(e_q e_j) f_p \otimes f_q \otimes f_j \ .$$

Comparing coefficients, we want to show that

$$\sum_i g(e_i e_j) f_i(e_p e_q) = \sum_i g(e_i e_j) f_j(e_p e_q) \ .$$

Now we make use of the fact that $e_i e_j = \sum_n c^n_{i,j} e_n$ and $f_i(e_j) = \delta_{i,j}$. We have

$$\begin{aligned}\sum_i g(e_i e_j) f_i(e_p e_q) &= \sum_i \left(\sum_n c^n_{i,j} g(e_n) \right) c^i_{p,q} \\ &= \sum_n \left(\sum_i c^n_{i,j} c^i_{p,q} \right) g(e_n) \ .\end{aligned}$$

We also have

$$\begin{aligned}\sum_i g(e_i e_j) f_j(e_p e_q) &= \sum_i \left(\sum_n c^n_{p,i} g(e_n) \right) c^i_{q,j} \\ &= \sum_n \left(\sum_i c^n_{p,i} c^i_{q,j} \right) g(e_n) \ .\end{aligned}$$

But in fact we have

$$\sum_i c^n_{i,j} c^i_{p,q} = \sum_i c^n_{p,i} c^i_{q,j}$$

as this is exactly the relationship between structure constants that tells us that A is associative. So the desired equality follows at once and Δ is coassociative, as claimed.

For the counit relationship we want to show that

$$g = \sum_{i,j} g(e_i, e_j) \epsilon(f_i) f_j = \sum_{i,j} g(e_i, e_j) f_i \epsilon(f_j) \ .$$

. That is, we want to show that

$$\sum_j g(e_1 e_j) f_j = \sum_i g(e_i e_1) f_i \ .$$

Indeed, uing the structure constants, we have $g(e_1 e_j) = c^n_{1,j} g(e_n)$ and $g(e_j e_1) = c^n_{j,1} g(e_n)$. It is not hard to see that the claimed identity follows from the fact that $c^n_{1,j} = c^n_{j,1}$.

Note the lack of symmetry here: in general, the dual space of an arbitrary infinite-dimensional algebra is not a coalgebra. This is a sign that algebras and coalgebras differ in some substantial ways. More generally, we have the so-called *Fundamental Theorom of Coalebgras*:

Theorem 16

If C is a coalgebra then every $x \in C$ is an element of some finite dimensional subcoalgebra of C. In particular, C is the union (and hence the sum) of its finite dimensional subcoalgebras.

Proof Let $d \in C$. Let $\Delta^2 := \Delta \otimes 1 = 1 \otimes \Delta$. Then

$$\Delta^2(d) = \sum_{i,j} a_i \otimes b_{i,j} \otimes c_j,$$

where we are free to chose both $\{a_i\}$ and $\{c_j\}$ to be linearly independent, and we will do so.

Let $B \subset C$ be the subspace of C spanned by $\{b_{i,j}\}$. We want to show that B is a *subcoalgebra* of C – in other words, that $B \subseteq B \otimes B$.

Note that by coassociativty we have $1 \otimes \Delta \otimes 1 = 1 \otimes 1 \otimes \Delta$. This means that

$$\sum_{i,j} a_i \otimes \Delta(b_{i,j}) \otimes c_j = \sum_{i,j} a_i \otimes b_{i,j} \otimes \Delta(c_j) \ .$$

But we have chosen $\{a_i\}$ is a linearly independent set. So this previous equation tells us that

$$\sum_j \Delta(b_{i,j}) \otimes c_j = \sum_j b_{i,j} \otimes \Delta(c_j)$$

for all $i \in I$.

It follows then that

$$\sum_j \Delta(b_{i,j}) \otimes c_j \in B \otimes C \otimes C \ ,$$

and as $\{c_j\}$ is a linearly independent set it follows that $\Delta(b_{i,j}) \in B \otimes C$ for all $i \in I, j \in J$.

Symmetrically, from the fact that $1 \otimes \Delta \otimes 1 = \Delta \otimes 1 \otimes 1$, we can show that $\Delta(b_{i,j}) \in C \otimes B$ for all $i \in I, j \in J$.

Hence it follows that

$$\Delta(b_{i,j}) \in (B \otimes C) \cap (C \otimes B) = B \otimes B \ ,$$

where for this last equality we use the general fact that if U and V are vector spaces with $U \subseteq V$ then $(U \otimes V) \cap (V \otimes V) = U \otimes U$. This completes the proof.

No similar result holds for algebras. For example, consider the algebra $\mathbb{K}[x]$ of polynomials in one variable. This is an infinite dimensional algebra generated by the element x. But this then means that x cannot be contained in any finite dimensional subalgebra.

Although we will not define them here, there are dual concepts of all the algebra conceptswe have described so far. There are coideals, cokernels, comodules and so on.

Some examples

- The *matrix coalgebra* has elements the set $M_n(\mathbb{K})$ of $n \times n$ matrices. This vector space has basis $\{E_{i,j}\}$ where $E_{i,j}$ is the matrix whose entries are all zero except for a 1 in position (i, j). A coalgebra structure on this vector space is given on basis elements by

$$\Delta(E_{i,j}) = E_{i,j} \otimes E_{i,j}$$
$$\epsilon(E_{i,j}) = \delta_{i,j} \ .$$

  where $\otimes$ is the Kronecker matrix product and $\delta_{i,j}$ is the Kronecker delta, that is

$$\delta_{i,j} = \begin{cases} 1 & \text{if} x = y \\ 0 & \text{if} x \neq y \end{cases} \ .$$

- If G is a (finite) group then $\epsilon(g) = 1$ and $\Delta(g) = g \otimes g$ for all $g \in G$. Elemebts of other coalgebras that have this second property are correspondingly called *grouplike elements*.

- If $\mathcal{P}$ is a partially ordered set, let $[x, y] := \{z \in \mathcal{P} \mid x \leq z \leq y\}$. The *incidence coalgebra* is the $\mathbb{K}$-vector space with basis $\{[x, y] \mid x, y \in \mathcal{P}\}$ and comultiplication and counit given by

$$\Delta([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y]$$
$$\epsilon([x, y]) = \delta_{x,y} \ ,$$

  where $\delta_{x,y}$ is the Kronecker delta defined above.

- There is a coalgebra structure on $U(\mathfrak{sl}_2)$, given by – for any $z \in \{e, f, h\}$ – $\epsilon(z) = 0$ and $\epsilon(z) = 1 \otimes z + z \otimes 1$. Elements of other coalgebras that have this second property are called *primitive*.

- The ring of polynomials $k[x]$ can be given a coalgebra structure. We have $\epsilon(x) = 0$ and

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

  .

- Similarly, for a quiver Q we can define a path coalgebra. The counit of any non-trivial path is zero while the comultiplication of a path is the sum of all pairs of paths whose concatenation gives that path. In fact, the previous example is just a special case of this for the quiver with one vertex and one edge labelled x.
- We previously defined an algebra structure on the infinite dimensional space $T(V)$. This space can also be given the structure of a coalgebra. Let $\mathcal{R}_{n,p} \subset S_n$ be the set of all permutations σ with the property that

  $$\sigma(1) < \sigma(2) < \cdots < \sigma(p)$$

  and

  $$\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n) .$$

  . We call this the set of (n,p) *riffle shuffles*. For $v_1 \otimes \cdots \otimes v_n \in T(V)$ we now define

  $$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{p=0}^{n} \sum_{\sigma \in \mathcal{R}_{n,p}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)} \otimes v_{\sigma(p+1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

  We define the counit by $\epsilon(1) = 1$ and otherwise

  $$\epsilon(v_1 \otimes \cdots \otimes v_n) = 0 .$$

  Note that the elements $v \in V$ are primitive in the sense defined earlier: $\Delta(v) = 1 \otimes v + v \otimes 1$.

Bialgebras and Hopf algebras

Similarly to how we motivated an algebra as being both a vector space and a module in a compatible way, a bialgebra is both an algebra and a coalgebra in some compatible way.

In order to define the right notion of compatibility, we return to the notion of an algebra homomorphism. First, recall that if A is an algebra then so is $A \otimes A$. Something very similar holds for coaelgbras.

Proposition 17

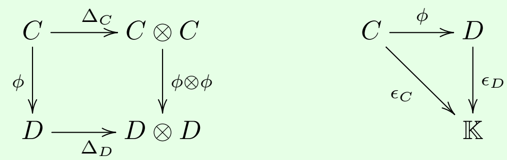
Let (C, Δ_C, η_C) and $(D, \Delta_C, \epsilon_D)$ be two coalgebras over a field $\mathbb{K}$. Then the tensor product $C \otimes D$ has a natural coalgebra structure given by

$$\Delta_{C\otimes D} := (1_C \otimes \tau \otimes 1_D) \circ (\Delta_C \otimes \Delta_D)$$
$$\eta_{C\otimes D} := \epsilon_C \otimes \epsilon_C .$$

Equally, there is a dual notion of coalgebra homomorphism::

Definition 18

Given two algebras C and D, an algebra homomorphism is a linear map $\phi : D \to D$ such that the diagrams below commute:



Motivated by the definition of an algebra morphism and a coalgebra morphism, we now have the right notion of compatibility:

Definition 19

A *bialgebra* is a $\mathbb{K}$-vector space B together with linear maps

$$\mu : B \otimes B \to B \ ,$$

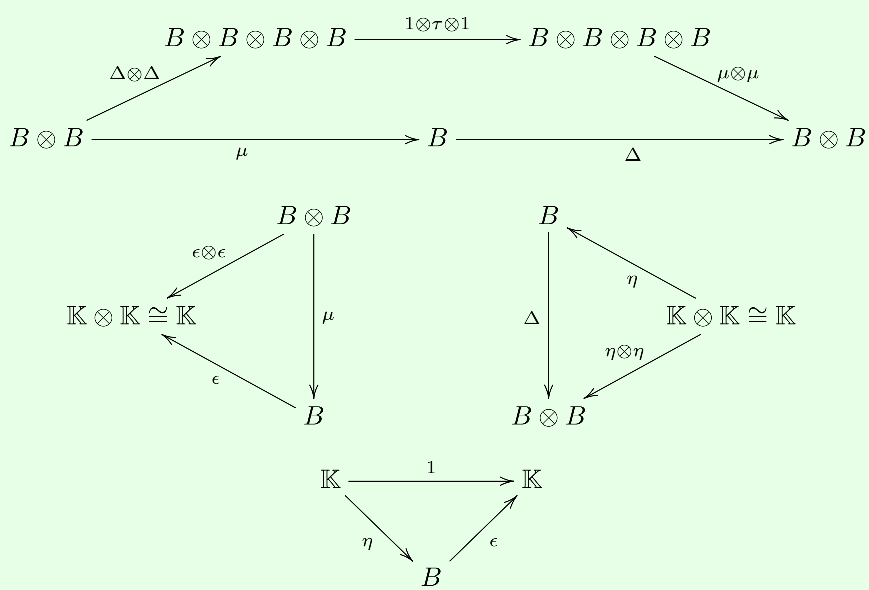
$$\eta : \mathbb{K} \to B \ ,$$

$$\Delta : B \to B \otimes B$$

and

$$\epsilon : B \to \mathbb{K}$$

such that (A, μ, η) is an algebra, (A, Δ, ϵ) is a coalgebra and the following diagrams commute:



where $\tau : B \otimes B \to B \otimes B$ is the linear map defined by $\tau(x \otimes y) = y \otimes x$ for all $x, y \in B$.

You can think of the diagrams as either telling us that Δ and ϵ are algebra morphisms or equivalently that μ and η are coalgebra morphisms. To move from one perspective to another, we simply rotate the diagrams in the obvious way.

Recall that the reason we were led to discuss coalgebras was the representation theory of algebras. In particular, we are trying to find algebras which share some of the nice represention theory properties of group algebras.

One further such nice property of group algebras comes from the fact that each element of the basis is invertible.

Definition 20

A *Hopf algebra* is a $\mathbb{K}$-vector space H together with linear maps

$$\mu : H \otimes H \to H \ ,$$

$$\eta : \mathbb{K} \to H \ ,$$

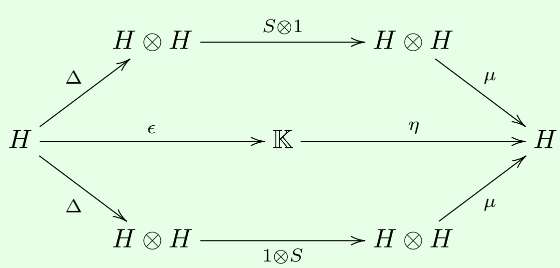
$$\Delta : H \to H \otimes H \ ,$$

$$\epsilon : H \to \mathbb{K}$$

and

$$S : H \to H$$

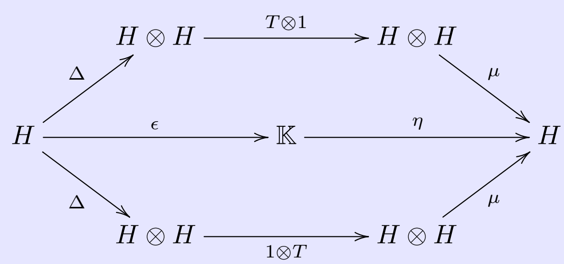
such that $(H, \mu, \eta, \Delta, \epsilon)$ is a bialgebra and the following diagram commutes:



Not every bialgebra can be given the structure of a Hopf algebra, but when this is possible there is always only one way to do it. This is equivalent to the fact that if a monoid has an identity element (ie if it is a group) then that identity element must be unique (and the proof of both results is similar).

Proposition 21

Suppose that $(H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra and that $T : H \to H$ is a linear map such that the diagram below commutes (i.e. suppose T is an antipode).



Then $T = S$.

Proof Let $x \in H$. Then $\Delta(x) = \sum_{i \in I} a_i \otimes b_i$ and we can choose $\{b_i\}$ such that they form a linearly independent set. By the diagram above (and the similar diagram defining an antipode) we have

$$\sum_{i \in I} S(a_i) b_i = \eta(\epsilon(x)) = \sum_{i \in I} T(a_i) b_i \ ,$$

which implies that, because the $\{b_i\}$ are linearly independent, we must have $S(a_i) = T(a_i)$ for all $i \in I$.

Since S and T are linear maps, it follows that

$$\begin{aligned} S(x) &= S\left(\sum_{i \in I} \epsilon(b_i) a_i\right) \\ &= \sum_{i \in I} \epsilon(b_i) S(a_i) \\ &= \sum_{i \in I} \epsilon(b_i) T(a_i) \\ &= T\left(\sum_{i \in I} \epsilon(b_i) a_i\right) \\ &= T(x) \ , \end{aligned}$$

and since this is true for an arbitrary x it follows that $S = T$.

Finally, we can put everything together and start talking about the representation theory of Hopf algebras. We only state a few initial definitions and results here but we will return to this topic in the next part of this series.

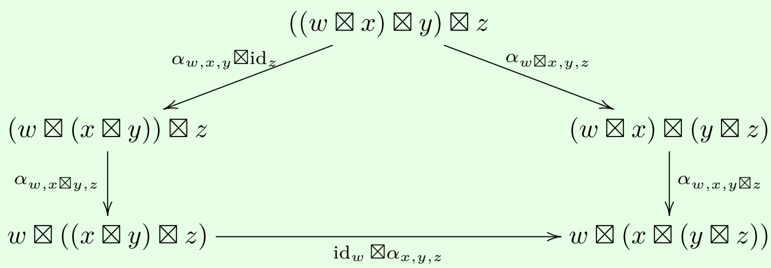
Definition 22

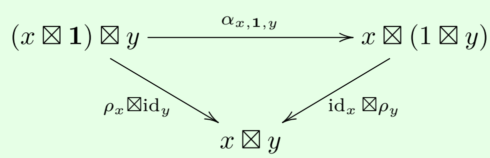
By a monoidal category we mean a tuple $(\mathcal{C}, \boxtimes, \mathbf{1}, \alpha, \lambda, \rho)$ where

- $\mathcal{C}$ is a category with objects $\mathrm{Obj}(\mathcal{C})$ and arrows $\mathrm{Arr}(\mathcal{C})$
- $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called the tensor product

- $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$ is an object in $\mathcal{C}$ called the unit object
- $\alpha : (- \boxtimes -) \boxtimes - \xrightarrow{\cong} - \boxtimes (- \boxtimes -)$ is a natural isomorphism called the associator with components $\alpha_{x,y,z} : (x \boxtimes y) \boxtimes z \to x \boxtimes (y \boxtimes z)$
- $\lambda : (\mathbf{1} \boxtimes -) \xrightarrow{\cong} -$ is a natural isomorphism called the left unitor with components $\lambda_x : 1 \boxtimes x \to x$
- $\rho : (- \boxtimes \mathbf{1}) \xrightarrow{\cong} -$ is a natural isomorphism called the right unitor with components $\rho_x : x \boxtimes 1 \to x$

such that the following diagrams commute for any $x, y, z, w \in \mathrm{Obj}(\mathcal{C})$:





Proposition 23

Let B be a bialgebra and let $B-\mathrm{Mod}$ denote the category of left B-modules. Then $B-\mathrm{Mod}$ is a monoidal category.

Proof In fact in this case all three isomorphisms α, λ and ρ are simply the identity map, so we can omit them here.

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two B-modules. Then the vector space $\mathcal{U} \otimes \mathcal{V}$ naturally has the structure of a $B \otimes B$ algebra: for any $x \otimes y \in B \otimes B$, $u \in \mathcal{U}$ and $v \in \mathcal{V}$ we have

$$(x \otimes y) \cdot (u \otimes v) = (x \cdot u) \otimes (y \cdot v) \ .$$

Now we can use the fact that $\Delta : B \to B \otimes B$ is an algebra morphism to give $U \otimes V$ the structure of a B-module. For any $x \in B$ we have

$$x \cdot (u \otimes v) = \Delta(x) \cdot (x \otimes y) = (\sum x_{(1)} \otimes x_{(2)}) \cdot (u \otimes v) = \sum (x_{(1)} \cdot u) \otimes (x_{(2)} \cdot v) \ .$$

We can call this new B-module $\mathcal{U} \boxtimes \mathcal{V}$. That this operation satisfies the required associativty axioms follows at once from the fact that Δ is coassociative.

For the unit object we use the fact that $\epsilon : B \to \mathbb{K}$ is a algebra homomorphism. This makes $\mathbb{K}$ a B-module and the axioms relating ϵ to the other structure maps of B ensure this module has the required properties.

Definition 24

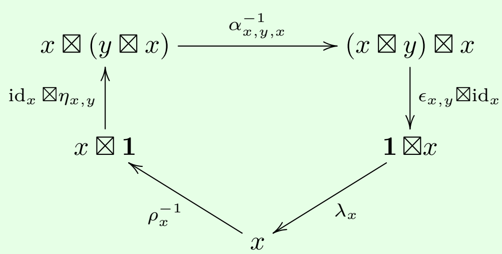
Let $(\mathcal{C}, \boxtimes, \mathbf{1}, \alpha, \lambda, \rho)$ be a monoidal category and let $x, y \in \mathrm{Obj}(\mathcal{C})$. We say that x is *left dual* to y – and conversely that y is *right dual* to x – if there exist arrows

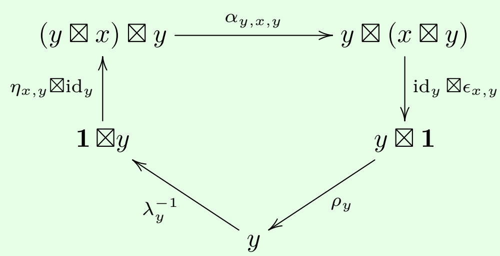
$$\eta_{x,y} : \mathbf{1} \to y \boxtimes x$$

and

$$\epsilon_{x,y} : x \boxtimes y \to \mathbf{1}$$

such that the following diagrams commute:





If $x^\star \in \mathrm{Obj}(\mathcal{C})$ is both left dual and right dual to x we say simply that it is dual to x. If every object in a monoidal category has a dual then we say that the monoidal category is rigid.

Proposition 25

Let H be a Hopf algebra and let $H - \mathrm{mod}$ denote the category of finite dimensional (as vector spaces over $\mathbb{K}$) left H-modules. Then $H - \mathrm{mod}$ is a rigid monoidal category.

Proof Clearly $H - \mathrm{mod}$ is a monoidal category because H is a bialgebra. What we have to show is that this category is rigid.

For any algebra A, if M is a (right) A-module then the dual $M^\star := \mathrm{Hom}(M, \mathbb{K})$ is a left A-module. If $a \in A$, $m \in M$ and $\psi : M \to \mathbb{K}$, then we define a left action of A on $M^\star$ by

$$(a \cdot \psi)(m) := \psi(m \cdot a) \; .$$

This operation actually extends to a duality between the category of left modules and the category of right modules.

For H a Hopf algebra, the existence of the antipode $S : H \to H$ gives us a second way of translating between left and right modules. IF M is a left H-module, then let $\overline{M}$ be the right module with the action of A on M defined by

$$m \cdot a := S(a) \cdot m \ .$$

Combining these two notions of duality gives us the duals we want. If $\mathcal{U}$ is any finite dimensional left H-module, then $(\overline{\mathcal{U}})^\star$ is also a (finite dimensional) left H-module. That this module has the required properties is left as an exercise.

Note that if $\mathcal{U}$ is a finite dimensional H-module with basis $\{e_i\}$ and $\{f_i\}$ is the corresponding dual basis of $(\overline{\mathcal{U}})^\star$ then the maps $\epsilon_{\mathcal{U},(\overline{\mathcal{U}})^\star}$ and $\eta_{\mathcal{U},(\overline{\mathcal{U}})^\star}$ – called *evaluation* and *coevaluation* respectively – are given explicitly by

$$\epsilon_{\mathcal{U},(\overline{\mathcal{U}})^\star}(f_i \otimes e_j) = \langle f_i, e_j \rangle = \delta_{i,j}$$

and

$$\eta_{\mathcal{U},(\overline{\mathcal{U}})^\star}(1) = \sum_i f_i \otimes e_i \ .$$

We will consider the category of modules of a Hopf algebra in more detail in later instalments in this series.

Some examples

- As stated in the motivation, the antipode for any group algebra $\mathbb{K}G$ is the map that sends g to g^{-1} for every $g \in G$. Indeed, $\mathbb{K}M$ is actually a bialgebra for any monoid M. An antipode for this bialgebra – making it a Hopf algebra – exists if and only if every element of M is invertible: that is, if M is in fact a group.

- We have given algebra and coalgebra structures for the vector space with basis the set of all $n \times n$ matrices. These two structures are in fact compatible, making the vector space a bialgebra. However it can be shown that this bialgebra has no antipode.

- We also saw previously that the tensor space $T(V)$ can be given both an algebra and a coalgebra structure. These structures are also compatible, making $T(V)$ a bialgebra. Furthermore, an antipode for this bialgebra is given by $S(1) = 1$ and

  $$S(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^m v_m \otimes v_{m-1} \otimes \cdots v_1$$

  meaning that $T(V)$ is actually a Hopf algebra. Moreover, if B is any bialgebra with n primitive elements, then there is a morphism from $T(\mathbb{K}^n) \to B$ which maps each of the he canonical basis elements $v_1, v_2, \ldots, v_n \in \mathbb{K}^n$ to a distinct primitive element of B. This means that, if U is a B-module for any bialgebra B, it is also a module for some tensor space algebra (using the construction given earlier).

- For $U(\mathfrak{sl}_2)$ the antipode is defined by $S(e) = -e$, $S(f) = -f$ and $S(h) = -h$. It is straightforward to check that this linear map has the required properties. For example,

$$\begin{aligned}(\mu \circ (S \otimes 1) \circ \Delta)(e) &= (\mu \circ (S \otimes 1))(1 \otimes e + e \otimes 1) \\ &= \mu(1 \otimes e - e \otimes 1) \\ &= e - e \\ &= 0 \\ &= \eta(0) \\ &= \eta(\epsilon(e)) \,.\end{aligned}$$

- Staying with the algebra $U(\mathfrak{sl}_2)$, we can now give an explicit example of how the coproduct lets us build up new representations. Since $\Delta(x) = 1 \otimes x + x \otimes 1$ for each of the primitive elements e, f, h, we can define a new representation $\rho_{1,1}$ by

$$\begin{aligned}\rho_{1,1}(x) &= \rho_1(1) \otimes \rho_1(x) + \rho_1(x) \otimes \rho_1(1) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \rho_1(x) + \rho_1(x) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,,\end{aligned}$$

where on the last line $\otimes$ denotes the Kronecker product. Direct calculation then gives us that

$$\rho_{1,1}(e) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\rho_{1,1}(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$
$$\rho_{1,1}(h) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \,.$$

The reader can check that this is indeed a valid representation of the algebra. In fact, for

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

further calculation gives us that

$$
\begin{aligned}
T^{-1}\left((\rho_2 \oplus \rho_0)(xe + yf + zh)\right) T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 9 \end{pmatrix}^{-1} \left(\begin{array}{ccc|c} 2z & 2x & 0 & 0 \\ 2y & 0 & x & 0 \\ 0 & y & -2z & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 9 \end{pmatrix} \\
&= \begin{pmatrix} 2z & x & x & 0 \\ y & 0 & 0 & x \\ y & 0 & 0 & x \\ 0 & y & y & -2z \end{pmatrix} \\
&= \rho_{1,1}(xe + yf + zh) \ ,
\end{aligned}
$$

and so the new representation $\rho_{1,1}$ is equivalent to the direct sum of two of the other representations we introduced previously. This is the first hint of a much more general result called the *Clebsch-Gordan decomposition formula* which we might return to later.

- An algebra A is called commutative if, for every $x, y \in A$, we have

  $$\mu(x \otimes y) = xy = yx = \mu(y \otimes x) \ .$$

  Similarly a coalgebra C is cocommutative if, for every $x \in C$, we have

  $$\Delta(x) := \sum x_{(1)} \otimes x_{(2)} = \sum x_{(2)} \otimes x_{(1)} \ .$$

  Most of the bialgebras we have seen so far are either commutative or cocommutative or both. For example, a group algebra is commutative exactly when the underlying group is abelian, but is cocommutative no matter the group. *Sweedler's Hopf algebra* is the four dimensional algebra with generators x, g, g^{-1} subject to the relations

  $$
  \begin{aligned}
  x^2 &= 0 \\
  g^2 &= 1 \\
  gx &= -xg \ .
  \end{aligned}
  $$

  The coproduct is given by

  $$
  \begin{aligned}
  \Delta(g) &= g \otimes g \\
  \Delta(x) &= 1 \otimes x + x \otimes g \ ,
  \end{aligned}
  $$

  the counit is given by

  $$
  \begin{aligned}
  \epsilon(g) &= 1 \\
  \epsilon(x) &= 0 \ ,
  \end{aligned}
  $$

  and the antipode is given by

  $$
  \begin{aligned}
  S(g) &= g^{-1} \\
  S(x) &= -xg^{-1}
  \end{aligned}
  $$

Sweedler constructed this algebra in 1969 as an example of a Hopf algebra that was neither commutative nor cocommutative. We will see many more examples of such Hopf algebras in the next post in this series.