Bismuth-Gueant-Pu 2019: Portfolio choice … under drift uncertainty

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A very mathy paper with detailed derivations. Useful for getting comfortable with cont time Bayesian learning in portfolio-related problems.

Intro

Historical background: Markowitz 1952 studies one-period portfolio choice problem in mean-variance framework. Then, CAPM was introduced based on this idea. Samuelson and Merton then generalized Markowitz problem to multi-period consumption/investment problem (in discrete and continuous time, respectively).

Specifically, Merton used PDE techniques to characterize optimal consumption and portfolio (investment) cohice processes. Extensions were introduced such as transaction costs & credit constraints.

Major advances to solve Merton's problem in full generality by Karatzas et al using martingale methods. This work itself was extended and we can use martingale method to solve Merton's problem for almost any smooth utility function.

Prop1: under $\mathbb{Q}$, μ is independent of W_t^Q, $\forall t \in [0, T]$.

Proof: we know that μ is independent of W_t under $\mathbb{P}$. We try to use this condition to show that $\forall a, b$, we have $\mathbb{E}^{\mathbb{Q}}[e^{ia\mu + ibW_t^{\mathbb{Q}}}] = \mathbb{E}^{\mathbb{Q}}[e^{ia\mu}]\mathbb{E}^{\mathbb{Q}}[e^{ibW_t^{\mathbb{Q}}}]$.

We first use some algebraic manipulations to show that $\mathbb{E}^{\mathbb{Q}}[e^{ia\mu+ibW_t^Q}] = \mathbb{E}[e^{ia\mu}]e^{-\frac{t}{2}b^2}$.

then recall $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\alpha(\mu)W_T - \frac{1}{2}\alpha(\mu)^2 T}$, where $\alpha(\mu) \equiv \frac{\mu - r}{\sigma}$.

Thus, $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mu] = 1$. We use this to show that $\mathbb{E}^{\mathbb{Q}}[e^{ia\mu}] = \mathbb{E}[e^{ia\mu}]$.

Then, we recall that the standard normal distribution has charactersitic function $e^{-\frac{t^2}{2}}$. So $e^{\frac{t}{2}b^2} = \mathbb{E}^{\mathbb{Q}}[e^{ibW_t^{\mathbb{Q}}}]$. Thus we are done.

Now we can start deriving an expression for $\beta_t \equiv \mathbb{E}[\mu|\mathcal{F}_t^S]$. And an expression for the dynamics of (β_t).

1. $\beta_t = \Sigma G(t, Y_t) + r$
2. $d\beta_t = \Sigma D_y G(t, Y_t)(\sigma d\hat{W}_t)$

Optimal Portfolio Choice

note that this is a Merton portfolio choice problem, the "strategy" here refers to the shares of stocks held, not the trading speed as in A&C.

Given strategy $(M_t)_{t\in[0,T]}$, the portfolio value process $(V_t)_{t\in[0,T]}$ satisfies:

$$dV_t = (M_t(\mu - r) + rV_t)dt + \sigma M_t dW_t$$

Note also that V_t here refers to the portfolio value at t, not the market volume V_t^i later when we consider the A&C type problem.

We rewrite portfolio value process as:

$$dV_t = (M_t(\beta_t - r) + rV_t)dt + \sigma M_t d\hat{W}_t$$

we can then plug in the expression for β_t.

We also rewrite dY_t (recall $Y_t \equiv log(S_t)$) using β and $d\hat{W}$.

Finally, starting at (t, y), we could express $Y_s^{t,y}$ and $V_s^{t,V,y,M}$ using the integram forms.

Define value function $v(t, V, y)$:

$$v(t, V, y) \equiv \sup_{(M_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}[-e^{-\gamma V_T^{t,V,y,M}}]$$

Note that here V is not value function, but the "wealth" process (of the portfolio), while y is the current log stock price.

The associated HJB is:

$\partial_t u + rV\partial_V u + ... = 0$, with terminal condition $u(T, V, y) = -e^{-\gamma V}$.

To solve this HJB, use an ansatz: $u(t, V, y) = -e^{-\gamma(e^{r(T-t)V + \phi(t,y)})}$

Plugging in, we get a simple, linear parabolic PDE, and can use the classic Feynman-Kac representation to write a strong solution of it.

Then, paper gives derivation of similar results for the case of CRRA agents.

Optimal portfolio choice in Gaussian case: two routes

again, solving optimal portfolio choice problem boils down to solving linear parabolic PDEs in the CARA & CRRA case. **one case for which these PDEs have closed form solutions is when the prior is Gaussian**.

There are two routs to solve the problems with PDEs: using β as state var, or using y. (again, recall y is log stock price)

First, we can show that under Gaussian prior, we have simplified expression for β and its dynamics:

1. define $\Gamma_t \equiv (\Gamma_0^{-1} + t\Sigma^{-1})^{-1}$
2. $\beta_t = ...$
3. $d\beta_t = ...$

classic result is that the posterior distro of μ given $\mathcal{F}_t^S$ is $N(\beta_t, \Gamma_t)$. Note that the covariance matrix process $(\Gamma_t)_t)$ is deterministic.

the problem can be written with two sets of state vars:

1. (y, V)
2. (β, V)

We can consider the previous optimization problem as being described by:

1. $dY_t = ...$
2. $dV_t = ...$

or,

1. $d\beta_t = ...$
2. $dV_t = ...$

Because $G(t, \cdot)$ is affine in y for all $t \in [0, T]$ in the Gaussian case, we see from the Feynman-Kac representation that $\forall t$, $\phi(t, \cdot)$ is a polynomial of degree 2 in y. But looking for this polynomial using PDE is cumbersome.

The main reason is that β is in fact a more natural variable to solve the problem, than y. In fact, the best ansatz if we try to solve the problem using state (y, V) is:

$\phi(t, y) = a(t) + \frac{1}{2}G(t, y)'B(t)G(t, y).$

plugging in this form to HJB we get system of linear ODEs for $a(t)$ and $B(t)$.

Now, solve the problem using β as a state var. define value function $\tilde{v}(t, V, \beta)$ via:

$\tilde{v}(t, V, \beta) \equiv \sup_{M \in \tilde{\mathcal{A}}_t} \mathbb{E}[-e^{-\gamma V_T^{t,V,\beta,M}}]$

where:

1. $\beta_s^{t,\beta} = \beta + \int_t^s ...$
2. $V_s^{t,V,\beta,M} = V + \int_t^s ...$

Then again, formulate HJB with terminal condition, make ansatz, and derive the associated system of linear ODEs.

Online learning & execution costs

Present a general optimization problem, derive HJB, derive a simpler PDE using ansatz. Then focus on special cases.

Again, here I consider the special case of a single risky asset, along with a bond with return $r = 0$.

The risky asset has price dynamics: $dS_t = \mu dt + \sigma dW_t^i$. μ is unknown, with prior distro denoted by mu_μ.

Again, let $\beta_t \equiv \mathbb{E}[\mu|\mathcal{F}_t^S]$

Theorem 8: define function $F(t, S) \equiv \int_{\mathbb{R}} e^{z\frac{S-S_0-\frac{t}{2}z}{\sigma^2}} m_\mu(dz)$, this function is well-defined. Let's define another function G via: $G \equiv \frac{\nabla_S F}{F}$, then $\beta_t = \sigma^2 G(t, S_t)$.

Then, define $(\hat{W}_t)_{t \in \mathbb{R})+}$ by $\hat{W}_t \equiv W_t + \int_0^t \frac{\mu - \beta_s}{\sigma} ds$. We can show that $(\hat{W}_t)_{t \in \mathbb{R})+}$ is a BM adapted to $(\mathcal{F}_t^S)_{t \in \mathbb{R}_+}$, and $dS_t = \beta_t dt + \sigma d\hat{W}_t = \sigma^2 G(t, S_t)dt + \sigma d\hat{W}_t$.

Now, we use inventory level q and cash level X as state variables.
1. $q_t = q_0 + \int_0^t v_s ds$ 2. $dX_t = -v_t S_t dt - V_t L(\frac{v_t}{V_t})dt$

Here (V_t) is a **deterministic process modelling the market volume** for the risky asset. L models execution costs, and satisfies certain properties like convexity. (In Almgren-Chriss, $L(y) = \eta y^2$)

Writing all the processes in the integral form:

1. $X_s^{t,x,S,v} = x + \int_t^s (-v_t S_\tau^{t,S} - V_\tau L(\frac{v_\tau}{V_\tau}))d\tau$

(starting at time t, when stock price is at S, agent's cash is at x, and agent uses trading stategy v)

2. $q_s^{t,q,v} = q + \int_t^s v_\tau d\tau$
3. $S_s^{t,S} = S + \int_t^s \sigma^2 G(\tau, S_\tau^{t,S})\tau + \int_t^s \sigma d\hat{W}_t$

Suppose agent's initial state is (x_0, q_0, S_0). The optimization problem is:
$\sup_{(v_t)_{t\in[0,T]}} \mathbb{E}[-e^{-\gamma(X_T^{0,x_0,S_0,v}+q_T^{0,q_0,v}S_T^{0,S_0}-\ell(q_T^{0,q_0,v}))}]$

1. often in liquidation problem assume $\ell(q) = \frac{1}{2}Aq^2$.

Introduce value function $\mathcal{V}(t, x, q, S)$, we can write a general HJB and terminal conditions.

Can also make an ansatz of the form of the value function:
$\mathcal{V}(t, x, q, S) = -e^{-\gamma(x+qS-\theta(t,q,S))}$, and then the HJB as well as the terminal conditions would be written in terms of this $\theta(t, q, S)$, which is not dependent on current wealth level x.

Note that in the general case, this simpler HJB is still not linear, and does not have closed form solutions generally (hence numerical schemes are required such as S-DFP may be of relevance). however, in the special case where:

1. Gaussian prior
2. execution costs and penalty functions are quadratic as in Almgren-Chriss

then solving the problem boils down to solve a system of ODEs. Specifically, we can show that in this special case,
$\theta(t, q, S) = a(t) + \frac{1}{2}b(t)G(t, S)^2 + G(t, S)(c(t)q + e(t)) + ...$, where
$a(t), b(t), ...$ solve a system of ODEs.

Finally, some numerical experiments. Main feature is **trend following**: buy stocks when stock price increases, sell stocks when price decreases, in a smooth manner.