Hanson 2007 Book – Background Info on Jump Diffusions

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Stochastic Jump Diffusions (Ch1-Ch3)

Overall I find the writing of this book to be sloopy and imprecise at times, and some crucial concepts could have been more carefully explained, This can be quite annoying if you are teaching yourself with this. While it certainly is much simpler than Oksendal-Sulem and avoids semimartingales all together, I really cannot recommend teaching yourself with this book. So I decided to stop using this book after reading the first 3 chapters. My summaries are below.

Ch1 Stochastic Jump and Diffusion Processes

Markov Process
SP $X(t)$ is a Markov process if the conditional probability satisfies: $\forall t \geq 0, \forall \Delta t \geq 0, \forall x \in \mathcal{D}_x$ (domain of state space), we have $Pr[X(t+\Delta t) = x|X(s), s \in [0,t]] = Pr[X(t+\Delta t) = x|X(t)]$.

Wiener Process

the standard Wiener process $W(t)$ has:

1. continuous path: $W(t^+) = W(t^-) = W(t)$
2. independent increments: $\Delta W(t_i) \equiv W(t_i + \Delta t_i) - W(t_i)$ are mutually independent for all t_i on non-overlapping time intervals
3. $W(t)$ is a **stationary process**: the distro of $\Delta W(t)$ is independent of t. Note that it is really difference stationary, should say Brownian motion has "stationary increments" to be precise.
4. $W(t)$ is Markov
5. $W(t)$ $N(0,t)$, so the density of $W(t)$ is $\phi_{W(t)}(w) \equiv \phi_n(w;0,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}$
6. $W(0) = 0$ with prob 1: $\phi_{W(0^+)}(w) = \delta(w)$
7. $Cov[W(t), W(s)] = \min[t,s]$

So, if we think of Brownian increments of equal time steps, $\Delta[W(t+i\Delta t)] \equiv W(t+(i+1)\Delta t) - W(t+i\Delta t)$, where $i = 0, 1, ...$ These are iid with normal distro:

$\phi_{\Delta W(t)}(w) = \phi_n(w;0,\Delta t) = \frac{1}{\sqrt{2\pi\Delta t}}e^{-\frac{w^2}{2\Delta t}}$

The book then refer to $dW(t) \equiv W(t+dt) - W(t)$ as "differential process", and when $dt > 0$, it has the same distro as $W(dt)$, which is normal with mean 0 and variance dt.

Non-differentiability of sample path: $\forall t > 0, \forall x > 0, Pr[\lim_{\Delta t \to 0^+}[|\frac{\Delta W(t)}{\Delta t}| > x]] = 1$

Poisson Processes

1. $P(t)$ has unit jumps: if jumps occurs at T_k, then $P(T_k^+) = P(T_k^- + 1$
2. $P(t)$ is right-continuous
3. $P(t)$ has independent increments: $\Delta P(t_i) \equiv P(t_i + \Delta t_i) - P(t_i)$ are mutually independent for all t_i on non-overlapping time intervals
4. $P(t)$ is a stationary process: distro of $\Delta P(t) \equiv P(t+\Delta t) - P(t)$ is independent of t. Again, the terminology should be "stationary increments".
5. $P(t)$ is Markov: $Pr[P(t+\Delta t) = k|P(s), s \le t] = Pr[P(t+\Delta t) = k|P(t)]$
6. $P(t)$ is Poisson distributed with mean $\mu = \lambda t$ and variance $\sigma^2 = \lambda t$: $\Phi_{P(t)}(k; \lambda t) = Pr[P(t) = k] \equiv p_k(\lambda t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$. Here p_k denotes the probability of the Poisson RV being equal to k, not some parameter.
7. $P(0^+) = 0^+$ with probability 1: $p_k(0^+) = \delta_{k,0}$
8. $Cov[P(t), P(s)] = \lambda \min[t, s]$
9. $(P(t) - \lambda t)$ is a martingale

Thus as for BM, $\Delta[P(t + i\Delta t)] \equiv P(t + (i+1)\Delta t) - P(t + i\Delta t)$ are iid, and has the same discrete Poisson distribution as $P(\Delta t)$: $\Phi_{\Delta P(t)}(k; \lambda\Delta t) = Pr[\Delta P(t) = k] \equiv p_k(\lambda\Delta t) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^k}{k!}$.

As with BM, define $dP(t) \equiv P(t+dt) - P(t)$, and this has the same discrete distro as $P(dt)$, i.e., $\Phi_{dP(t)}(k; \lambda dt) = Pr[dP(t) = k] = p_k(\lambda dt) = e^{-\lambda dt}\frac{(\lambda dt)^k}{k!}$.

If we are to simulate $P(t)$, usually simulate time between jumps as we can show that $T_{k+1} - T_k$ has exponential distro: $\Phi_{\Delta T_j}(\Delta t) \equiv Pr[T_{j+1} - T_j \le \Delta t | T_j] = 1 - e^{-\lambda \Delta t}$.

Poisson 0-1 Jump law
As $\Delta t \to 0$, $\Delta P(t) = 1$ with probability $\lambda \Delta t$, otherwise no jumps. Other possibilities have probabilities that vanish quicker than these two. To be "precise", the book states it as:

1. As $\Delta t \to 0^+, Pr[\Delta P(t) = 0] = 1 - \lambda\Delta t + O^2(\lambda\Delta t)$
2. As $\Delta t \to 0^+, Pr[\Delta P(t) = 1] = \lambda\Delta t + O^2(\lambda\Delta t)$
3. As $\Delta t \to 0^+, Pr[\Delta P(t) > 1] = O^2(\lambda\Delta t)$

Then using this 0-1 jump law, we formally write $E[f(dP(t))] =_{dt} (1 - \lambda dt)f(0) + \lambda dt f(1)$

temporal/non-stationary Poisson process
time dependent jump rate: $\lambda = \lambda(t)$.

given the rate process $\lambda(t)$, define $\Lambda(t) \equiv \int_0^t \lambda(s)ds$, or in differential form, $d\Lambda(t) \equiv \lambda(t)dt$.

The **temporal Poisson process* has the following analogous results:

1. $\Phi_{dP(t)}(k; \lambda(t)dt) = Pr[dP(t) = k] = p_k(\lambda(t)dt) = e^{-\lambda(t)dt}\frac{(\lambda(t)dt)^k}{k!}$
2. $\Phi_{\Delta P(t)}(k; \Delta\Lambda(t)) = Pr[\Delta P(t) = k] = p_k(\Delta\Lambda(t)) = e^{-\Delta\Lambda(t)}\frac{(\Delta\Lambda(t))^k}{k!}$
3. $\Phi_{P(t)}(k; \Lambda(t)) = Pr[P(t) = k] = p_k(\Lambda(t)) = e^{-\Lambda(t)}\frac{(\Lambda(t))^k}{k!}$
4. $E[\Delta P(t)] = \Delta\Lambda(t)$
5. $Var[\Delta P(t)] = \Delta\Lambda(t) = \int_t^{t+\Delta t} \lambda(s)ds$
6. up to order dt, with prob $\lambda(t)dt$ we have $dP(t) = 1$, otherwise $dP(t) = 0$
7. inter-jump times is again exponentially distributed: $\Phi_{\Delta T_{j-1}|T_{j-1}}(\Delta t) = 1 - e^{-\int_{T_{j-1}}^{T_{j-1}+\Delta t} \lambda(t)dt}$

Ch2 Stochastic Integration for Diffusions

Jump diffusion SDE with initial conditions has the form:

1. $dX(t) = f(X(t), t)dt + g(X(t), t)dW(t) + h(X(t), t)dP(t)$
2. $X(0) = x_0$

This is a symbolic equation, it has no meaning until we specify the **methods of integration** for the **3 types of integrals**:

$X(t) = x_0 + \int_0^t f(X(s), s)ds + \int_0^t g(X(s), s)dW(s) + \int_0^t h(X(s), s)dP(s)$

Riemann Integration

1. use $I[f](t)$ to denote $\int_0^t f(s)ds$
2. the partition of the interval $[0, t]$ is index by $t_0 = 0, t_1, ..., t_n, t_{n+1} = t$. A total of $n + 1$ intervals. Denote δt_n as mesh size
3. on each subinterval, take an "approximation point" $t_i^* \equiv t_i + \theta_i\Delta t_i$, where $\theta_i \in [0, 1]$.
4. Define (constructively) $I[f](t) \equiv \lim_{n\to\infty, \delta t_n \to 0}[I_n^{(\theta)}[f](t)]$, where $I_n^{(\theta)}[f](t) \equiv \sum_{i=0}^n f(t_{i+\theta_i})\Delta t_i$
5. Because BM $W(t)$ is continuous with prob1, the integral of $f(W(t), t)$ wrt t can be defined via Riemann: $\int_0^t f(W(s), s)ds = \lim_{n\to\infty}[\sum_{i=0}^n f(W(t_i), t_i)\Delta t_i]$. Here we chose $\theta = 0$ but any θ is fine.

6. Even for $X(t)$ as solutions to jump-diffusion SDEs, can define $\int_0^t f(X(s), s)ds$ this way.
7. Stieltjes integral refers to a deterministic integration wrt the position on the path of $X(t)$. Define it (constructively) as: $\int_0^t f(X(s), s)dX(s) \equiv \lim_{n\to\infty}[\sum_{i=0}^n f(X(t_i + \theta\Delta t), t_i + \theta\Delta t)(X(t_{i+1}) - X(t_i))]$. This makes sense if $X(t)$ is continuous and BV.

Ito integration wrt $W(t)$

start with trying to define $I[W](t) \equiv \int_0^t W(s)dW(s)$. You would expect it may mimic deterministic case where $I^{((det))}[X](t) = \int_0^t X(s)dX(s) = \frac{1}{2}\int_0^t d(X^2)(s) = \frac{1}{2}(X^2(t) - X^2(0))$. But turns out to be not the case.

So we go back to a discrete approximation first, and we use Ito's choice of approximation ($\theta = 0$) to preserve independent increments:

$I_n^{(0)}[W](t) = \sum_{i=0}^n W(t_i)\Delta W(t_i) \equiv \sum_{i=0}^n W_i \Delta W_i$

After some algebra, we can show that $I_n^{(0)}[W](t) = \frac{1}{2}(W^2(t) - \sum_{i=0}^n (\Delta W_i)^2)$

Then the book calculated the expectation of this expression, which is 0, and claims that this suggest a reasonable form of stochastic integral to be: $I[W](t) = \frac{1}{2}(W^2(t) - t)$. I don't see why it seems natural, but certainly the sum of the squares converge to t in L2 sense, and in the end the conjecture is correct.

convergence in mean square The RV $I_n^{(0)}(t)$ converges in mean square to RV $I(t)$ if $E[(I_n^{(0)}(t) - I(t))^2] \to 0$, and we write it as $I(t) = \underset{n\to\infty}{\overset{ms}{\lim}}[I_n^{(0)}(t)]$, or we use the notation $\overset{ims}{=}$ which stands for **Ito mean square equals to**$.

It would seem here that whenever this $\overset{ims}{=}$ appears, to its right it should be a discrete approximation, for example, $t \overset{ims}{=} \sum_{i=0}^n (\Delta W_i)^2$. However, later on this $\overset{ims}{=}$ is used quite generously.

Then book proves the ms limit of the sum of squared Brownian increments is t, and using that, we can prove the ms limit of $I_n^{(0)}[W](t)$ is indeed $\frac{1}{2}(W^2(t) - t)$.

This gives a rigorous definition of the expression $\int_0^t W(s)dW(s)$, now we move to define Ito integral for more general integrands.

Ito mean square (ims) limit stochastic integral

For integrals of the form $I[g](t) \equiv \int_{t_0}^t g(W(s), s)dW(s)$, we first define its forward integration approximation as:

$I_n^{(0)}[g](t) \equiv \sum_{i=0}^n g(W(t_i), t_i)(W(t_{i+1}) - W(t_i))$,

then, denote $I^{(ims)}[g](t) = \underset{n\to\infty}{\overset{ms}{\lim}}[I_n^{(0)}[g](t)]$

If this limit exists, we define $I[g](t)$ to be this limit $I^{(ims)}[g](t)$. Note that we would need to require g to be bounded in the sense that $E[\int_{t_0}^t g^2(W(s),s)ds]$ is finite. Note also that $I[g](t)$ in principle have other evaluations, depending on our choice of "θ".

The book then summarizes the results so far, by claiming:

1. $\int_0^t (dW)^2(s) \overset{ims}{=} t$ (recall we said this notation $\overset{ims}{=}$ will be used quite generously later. This is one instance)
2. $\int_0^t W(s)dW(s) \overset{ims}{=} \frac{1}{2}(W^2(t)-t)$

The first expression is meant to say:

$\int_0^t (dW)^2(s) \overset{ims}{=} \underset{n\to\infty}{\overset{ms}{\lim}} [\sum_{i=0}^n (\Delta W)^2(t_i)] = t$

That is, sometimes we see a Ito integral to the left of $\overset{ims}{=}$, and to the right we have some term not involving n. The discrete approximation is then implicit – we approximate the Ito integral as the ms limit of a discrete forward approximation, which is then shown to equal to the RHS. It's hard for me to see the necessity of putting "ims" on top of the inequality, but I guess the book emphasizes that the symbolic stochastic integral doesn't have to be calculated in the Ito sense, using the forward difference. So we want to emphasize we are taking the Ito integral by writing "ims" on top.

It is indeed remarkable that you can take an arbitaray Brownian sample path, partition time interval into fine subintervals, and then calculate the sum of the square of Brownian increments for different terminal time. It turns out this sum as a function of terminal time resembles the linear function $y(t) = t$. Note it is just one arbitrary sample path, we are not empirically verifying the ms convergence.

Compare this the continuously differentiable case, the corresponding quadratic of a differential \$(dx)^{2}(t) would be negligible relative to terms of order dt:

$\int_0^t (dx)^2(s) \equiv \lim_{n\to\infty} [\sum_{i=0}^n (\Delta x_i)^2]$, and after some algebra we can show that this is equal to 0.

I find mathstackexchange has some nice discussion on $(dW)^2 = dt$. I'll summarize it here.

1. one can show $\sum_{j=1}^n g(B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}})^2$ converges to $\int_0^t g(B_s)ds$ as mesh size goes to 0
2. this is based on the fact that $B_t^2 - t$ is a martingale
3. It seems natural to **DEFINE** $\int_0^t g(B_s)dB_s^2 \equiv \lim_{n\to\infty} \sum_{j=1}^n g(B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}})^2$
4. thus, we have $\int_0^t g(B_s)dB_s^2 \equiv \int_0^t g(B_s)ds$, ergo, the heuristic rule

The second expression of the book ($\int_0^t W(s)dW(s) \overset{ims}{=} \frac{1}{2}(W^2(t) - t)$) gives the following expression by the book:

$W(t)dW(t) \underset{ms}{\overset{dt}{=}} \frac{1}{2}(d(W^2)(t) - dt)$. It reads "equal in dt-precision mean square. I don't see how this can be useful though.

fundamental theorem of Ito calculus

1. if $g(\cdot)$ is continuous, then $d[\int_0^t g(W(s))dW(s)] \overset{ims}{=} g(W(t))dW(t)$
2. if $G(\cdot)$ is C1, then $\int_0^t dG(W(s)) \overset{ims}{=} G(W(t)) - G(0)$

for exact derivatives, Ito stochastic integration and ordinary Riemann integration agree.

Now I have no idea what this is supposed to mean. First of all, the RHS of eq1. is not a well-defined random variable. Also, some book calls fundamenal theorem of Ito calculus to be Ito's lemma.

But as we mentioned before, $\overset{ims}{=}$ denotes discrete approximation, so for eq1 the discrete approx is $\Delta[\int_0^t g(W(s))dW(s)] = (\int_0^{t+\Delta t} - \int_0^t)g(W(s))dW(s)$.

Then, $\int_t^{t+\Delta t} g(W(s)dW(s) \approx g(W(t))\Delta W(t) \to g(W(t))dW(t)$, where we used continuity of g, W.

For eq2, the formal derivation is then: $LHS = \underset{n\to\infty}{\overset{ms}{\lim}}[\sum_{i=0}^n (G(W(t_{i+1})) - G(W(t_i)))]$.

Then we see that the terms cancel so that $LHS = \underset{n\to\infty}{\overset{ms}{\lim}}[\sum_{i=0}^n (G(W(t_{i+1})) - G(W(t_i)))] = G(W(t)) - G(0)$

In general, some assumptions are needed so that ms convergence works. The book calls it "i-PWCA", aka Piece-Wise-Constant Approximationsin the Ito Sense. This is consistent with more standard/modern treatment of Ito integral where we use a sequence of simple functions to approximate the integrand in the L2 sense.

Then, it is shown that if we assume ms integrability: $E[\int_{t_0}^t g^2(W(s), s)ds] < \infty$, we have $E[\int_{t_0}^t g(W(s), s)dW(s)] \overset{ims}{=} 0$. While it is kinda intuitive when we write the discrete approximation using forward difference, the rigorous proof is not trivial at all.

Ito isometry

1. $E[(\int_{t_0}^t g(W(s), s)dW(s))^2] \overset{ims}{=} \int_{t_0}^t E[g^2(W(s), s)]ds$
2. $E[(\int_{t_0}^t f(W(s), s)dW(s)) \int_{t_0}^t g(W(r), r)dW(r)] \overset{ims}{=} \int_{t_0}^t E[f(W(s), s)g(W(s), s)]ds$

Finally, this section establishes some other properties of Ito integral, such as linearity, additivity, continuity of sample paths. And demonstrated the heuristic rules such as $dtdW(t) = 0$.

Note that now we know that it means: $(dW)^3 = 0$ here is written as $(dW)^3(t) \underset{ms}{\overset{dt}{=}} 0$, and it means $\int_0^t (dW)^3(s) \overset{ims}{=} 0$.

Stratonovich Integral Recall Ito approximates integrand using $\theta = 0$. Now we do discrete approximation for general θ.

$I[W](t) = \int_0^t W(s)dW(s) \approx I_n^{(\theta)}[W](t) \equiv \sum_{i=0}^n W(t_{i+\theta})\Delta W_i$

We would then manipulate this sum to get familiar terms like $\sum(\Delta W_i)^2$, and new terms like $\sum(W(t_{i+\theta}) - W(t_i))^2$ which can be shown to converge ms to θt. In the end, we have:

$\int_0^t W(s)dW(s) \underset{ms}{\overset{\theta}{=}} I^{(\theta)}[W](t) = \frac{1}{2}W^2(t) - (\frac{1}{2} - \theta)t$

Stratonovich sets $\theta = \frac{1}{2}$. Note that when $\theta \neq 0$, the expectation of the integral in general is not 0.

Ch3 Stochastic Integration for Jumps

A more modern/standard treatment of this topic is here. The author is Nicolas Privault and he has extensive notes online about stochastic analysis and math fin.

Definition: $\int_0^t h(X(s), s)dP(s) \overset{ims}{=} \underset{n\to\infty}{\overset{ms}{\lim}} [\sum_{i=0}^n h(X(t_i), t_i)\Delta P(t_i)]$ (we require $E[\int_0^t h^2(X(s), s)ds]$ to be finite, and we need $Y(t) \equiv h(X(t), t)$ to satisfy the "i-PWCA" assumption, i.e., can be well approximated by simple functions)

Following the same "proof", we can show the "fundamental thm of Poisson jump calculus:

1. $d(\int_0^t h(P(s))dP(s)) \overset{ims}{=} h(P(t))dP(t)$
2. $\int_0^t d\mathcal{H}(P(s)) \overset{ims}{=} \mathcal{H}(P(t)) - \mathcal{H}(0)$

Again, I don't know exact what eq1 means. Its proof says:

1.1 $\Delta(\int_0^t h(P(s))dP(s)) = (\int_0^{t+\Delta t} - \int_0^t)h(P(s))dP(s)$

1.2 $\int_t^{t+\Delta t} h(P(s))dP(s) \approx h(P(t))\Delta P(t)$

1.3 $h(P(t))(P(t + \Delta t) - P(t)) \to h(P(t))dP(t)$

** now I think of it, perhaps the $\overset{ims}{=}$ in eq1 should be replaced with $\overset{dt}{=}$.**

Now, the book moves on to prove a special case: $I[P](t) \equiv \int_0^t P(s)dP(s) \overset{ims}{=} \frac{1}{2}(P(P-1))(t) \equiv I^{(ims)}[P](t)$

It seems like a heuristic argument based on "dt-calculus" is provided first, then the ms limit is rigorously proven.

Then this is generalized to: $\int_0^t h(P(s))dP(s) \overset{ims}{=} \sum_{k=0}^{P(t)-1} h(k)$

Note that if $P(t) = 2$, then the integral evaluates to $h(0) + h(1)$. This is because we are using forward approximation: at the time when the Poisson process jumps from 0 to 1, the contribution to the integral is $h(0)$, and when the Poisson process jumps from 1 to 2 (at some time before t), it contributes another term $h(1)$, and that's it.

Also note that Nicolas Privault seem to disregard the issue with $P(t) - 1$ and just write $\int_0^T N_t dN_t = \sum_{k=1}^{N_T} k = \frac{1}{2} N_T(N_T + 1)$. This seems to be the standard definition... elsewhere people talk about for a deterministic function u, $\int_0^t u(s)dN(s) = \sum_{k=1}^{N(t)} u(T_k)$.

Finally, we have the most general Poisson integral formula:

$\int_0^T h(X(s), s)dP(s) \overset{ims}{=} \sum_{k=1}^{P(t)} h(X(T_k^-), T_k^-)$,

unfortunately I don't know what T_k^- is supposed to mean and I don't know why we have $P(t)$ instead of $P(t) - 1$ as before. Book on p73 mentioned that it is "pre-jump time", as if it is an actual time instant. I thought $X(T_k^-)$ might mean the left limit of $X(\cdot)$ at T_k, but the book also uses expressions like $t_i + \Delta t \leq T_k^-$, as if it really is a particular constant.

update: I was reading Seydel's "Tools for Computational Finance" when it occurred to me that we do need to distinguish between pre- and post- jump values, since it is awkward to say that the jump size is 2 times the current state, for example, as the current state already incorporates the jump. So often, we say jump size is 2 times the pre-jump value of the current state.

Specifically, on p61 of Seydel, jump size is defined as $\Delta S = S_{\tau^+} - S_{\tau^-}$, where:

1. τ^+ denotes the infinitesmial instant immediately after jump
2. τ^- denotes the infinitesmial instant immediately before jump

Although I still feel like we could have just write τ^+ as τ.

Then some heuristic rules are derived:

1. $dtdP(t) \overset{dt}{=} 0$
2. $dP(t)dW(t) \overset{dt}{=} 0$
3. $(dP)^m(t) \overset{dt}{=} dP(t)$

And finally, **isometry** is proved. Letting $d\hat{P}(t) \equiv dP(t) - \lambda(t)dt$, we have:

1. $E[\int_{t_0}^t h(X(s), s)d\hat{P}(s)] \overset{ims}{=} 0$
2. $E[\int_{t_0}^t h_1(X(s), s)d\hat{P}(s) \int_{t_0}^t h_2(X(r), r)d\hat{P}(r)] \overset{ims}{=} \int_{t_0}^t E[h_1(X(s), s)h_2(X(s), s)]\lambda(s)ds$