Algebraic geometry (Autumn 2024), 1

Clark Barwick • 10 Sep 2024

So this autumn (starting 7 October), I'm teaching a course for first-year PhD students. It's algebraic geometry, which for me means the theory of schemes. I taught this once before, last spring. There are a few things I want to do different this time – some concepts that I didn't quite get across terribly well, some examples I didn't dig into deeply enough.

There are a zillion little choices you have to make when you teach a course, particularly one that has as many radical new ideas as algebraic geometry. [1] I try to make these choices through the lens of two questions:

1. What do I actually know how to communicate well? No one benefits from me stumbling through a discussion of something I don't understand well or even a story I understand but can't tell in an interesting way. The moduli space of K3 surfaces seems fascinating, but I know nothing nontrivial about it yet, so teaching a course with that as a centerpiece would be a stupid thing for me to do.
2. What does this particular crop of students need (or want) to know? Most of the students I'll be teaching are specializing in other areas of mathematics. If I were to spend a long time on examples over finite fields, I'd probably teach a course that would be of no use to anyone present.

Here I'll reflect some on these questions in the context of this course.

For whatever reason, I like to organize my thoughts about teaching around aphorisms. Today I have two. The first is about understanding what the word 'geometry' actually means in the phrase 'algebraic geometry'. The second is about understanding the role of the Zariski topology.

$\mathrm{geometry} = \mathrm{topology} + \mathrm{functions}$

This is a coarse statement, but it is broadly reflective of the attitude of the French school around Bourbaki in the late 50s and early 60s.

For example, when we talk about a manifold (smooth, say), what exactly are we talking about? You have a topological space X (Hausdorff, second-countable, paracompact), and then what do you have to add? One option is to say that you need a maximal smooth atlas (or an equivalence class of smooth atlases). Using such an atlas, you can now define smooth functions locally on X. If $U \subseteq X$, you get a ring (in fact an $\mathbf{R}$-algebra) $\mathcal{C}^\infty(U)$ of smooth functions on U. On the other hand, once you know what ‘smooth function’ means, you can reverse engineer an atlas.

So the data of a manifold can be expressed like this: it’s a topological space (satisfying some conditions perhaps) X along with a *sheaf of rings* $U \mapsto \mathcal{C}^\infty(U)$.

Well, almost. The stalk of this sheaf at a point $x \in X$ is not just a ring but a *local ring*. For every open neighborhood U of x, you have the ring map $\mathcal{C}^\infty(U) \to \mathbf{R}$ given by evaluation at x. These are all compatible, so they induce a ring map $\mathcal{C}^\infty_x \to \mathbf{R}$. The kernel is a maximal ideal $\mathfrak{m}_x \subset \mathcal{C}^\infty_x$ consisting of the germs of functions that vanish at x. The units of the stalk $\mathcal{C}^\infty_x$ are exactly the elements outside the maximal ideal; after all, if f is a smooth function on a neighborhood of x and $f(x) \neq 0$, then $1/f$ makes sense on a neighborhood of x. Thus the stalk is indeed a local ring.

The fact that the maximal ideal of a local ring is unique misleads one into thinking that locality is a *property* of a ring. It should really be considered as *structure*, because the maximal ideal isn’t preserved by every ring map. [2] One of the ways we try to remind ourselves of that is by writing a local ring as a triple $(R, \mathfrak{m}, k)$, where $\mathfrak{m}$ is the maximal ideal, and $k = R/\mathfrak{m}$ is the *residue field*. A *local ring homomorphism* $\phi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is a ring homomorphism such that $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$ – or, equivalently, one that induces a field extension $k \hookrightarrow \ell$.

So that’s the structure: a topological space X and a sheaf of rings whose stalks are local – a *locally ringed space*. And the claim is that being a smooth manifold is a particular *property* that a locally ringed space might possess. In other words, smooth manifolds and smooth maps between them form a full subcategory of the category of locally ringed spaces.

The property that defines this full subcategory is the fact that, locally, manifolds of the same dimension all look the same: they all look like euclidean space. In other words, a smooth manifold is a locally ringed space that is *locally isomorphic* to $(\mathbf{R}^n, \mathcal{C}^\infty)$.

Once you start thinking like this, it’s hard to stop. You can specify a style of geometry by just specifying the local models you like. If you take locally ringed spaces of the form $(\mathbf{C}^n, \mathcal{O}^{hol})$, then you arrive at the geometry of complex manifolds. And if you take locally ringed spaces of the form

$$\operatorname{Spec} R = ((\operatorname{Spec} R)_{zar}, \mathcal{O}_{\operatorname{Spec} R}),$$

then you arrive at the category of *schemes*!

This perspective also opens to door to generalizations that provide a much bigger universe of geometries. For example, in analytic geometry, the rings of functions are equipped with something like a topology (or condensed structure). In derived algebraic geometry, the rings of functions are equipped with a derived structure (a cdga or a ring spectrum). With various theories of stacks, one may need to replace the topological space with a site (or a topos).

In a first course on scheme theory, there's no need to get into these variations of course. Schemes are hard enough. However, I want to communicate the idea that this approach to geometry establishes a theme upon which many variations can be fruitfully created and explored.

But ok. The basic local models are affine schemes $\operatorname{Spec} R$. But how do we talk about these locally ringed spaces? The first step is to understand this Zariski topological space. For that, I have another aphorism.

The Zariski topology is a glorified poset

The Zariski topology on $\operatorname{Spec} R$ meets the requirements of a topology, but the intuitions one develops in a general topology course are not terribly useful for understanding it. It's (in)famously weird. In particular, the open sets are comically large, and as a result, it's rarely Hausdorff, but it's always quasicompact (*i.e.*, every open cover contains a finite subcover). It feels *pathological*.

But our aphorism is an effort to think about this a little differently. The claim is that the Zariski topology can be thought of in more combinatorial terms. I'm not going to give a definition of $(\operatorname{Spec} R)_{zar}$ here today; I'll do that in a later post. Rather, I want to think about the nature of the Zariski topology, and why it's so weird.

We can begin by reflecting on the notion of a *stratification* of a topological space. A stratification of X is sometimes defined as sequences of closed subspaces

$$X_{\leq 0} \subseteq X_{\leq 1} \subseteq \cdots \subseteq X$$

whose union is all of X. Some authors require some further conditions, but let's be relaxed about this. The n-th *stratum* is the subset $X_n = X_{\leq n} \setminus X_{\leq n-1}$, which is *locally closed*. For instance, you can consider projective space $\mathbf{CP}^n$, which is stratified via the hyperplanes at infinity:

$$\mathbf{CP}^0 \subset \cdots \subset \mathbf{CP}^n,$$

so that the strata are affine spaces.

In general, if X is stratified in this sense, we have a map $p\colon X \to \mathbf{N}$ that carries a point of X to the stratum it lives in. Now we can topologize $\mathbf{N}$ so that this map is continuous. [3] We say that a subset $U \subseteq \mathbf{N}$ is open iff it is an *upper set*: if $x \leq y$, then if $x \in U$, then $y \in U$. With that topology – the *Alexandrov topology* – giving a stratification on X is the same thing as giving a continuous map $p\colon X \to \mathbf{N}$.

Of course, there's nothing terribly special about $\mathbf{N}$ here. I could do the same thing with any poset P: endow it with the Alexandrov topology, define a P-stratification as a continuous map $p : X \to P$. You have a locally closed v-th stratum $X_v = p^{-1}(v)$ for any $v \in P$, and you also have closed subsets like

$$X_{\leq v} = \bigcup_{w \leq v} X_w = \{x \in X : p(x) \leq v\}.$$

This lets you think about the combinatorics of more complex kinds of stratifications, like the following stratification on the sphere S^n.

First, let's identify our poset P. Its elements will be the vertices of an n-cube:

$$P = \{e_i, -e_i : 1 \leq i \leq n\}$$

Order this set so that if $i \leq j$, then either of $-e_i, e_i$ is smaller than either of $-e_j, e_j$. Now define $p : S^n \to P$ in the following way: let i be the largest index such that $x_i \neq 0$, and set $p(x) = x_i/|x_i|e_i$.

By the way, this stratification is actually pretty important, and I want to return to it in another context. [4]

Now let's think about what happens when we contemplate *algebraically defined stratifications* of $\mathbf{C}^n$. Algebraic geometry, we are led to understand, is the study of solutions of polynomial equations. So if I have a collection $S = \{f_1, \ldots, f_k\}$ of polynomials in n variables, then I'm interested in the sets $Z(S) = \{x \in \mathbf{C}^n : f_1(x) = \cdots = f_k(x) = 0\}$. These are closed subsets of $\mathbf{C}^n$. And these are, like, the only ones we want to think about.

So an *algebraic stratification* of $X = \mathbf{C}^n$ consists of a finite poset P and a continuous map $p\colon X \to P$ (Alexandrov topology again) such that for every $v \in P$, the subset $X_{\leq v}$ is a subset of the form $Z(S_v)$ for some set S_v of polynomials. In fact, we might as well choose $S_v = V(X_{\leq v})$, the set of all polynomials that vanish at every point of $X_{\leq v}$.

For example, I could do something pretty simple, and let $P = \{0, \ldots, n\}$, and define $p(x)$ to be the smallest i such that $x_i \neq 0$. But I could also do something far more combinatorially complicated. When you can write a poset map $f\colon P \to Q$, then we think of a stratification $p\colon X \to P$ as *finer* than the resulting stratification $fp\colon X \to Q$.

Now here's a basic question you can ask yourself: is there a *finest*, or *universal* algebraic stratification of X? That is, is there an algebraic stratification $p^{uni}\colon X \to P^{uni}$ such that for every algebraic stratification $p\colon X \to P$, there is a unique poset map $f\colon P^{uni} \to P$ such that $p = fp^{uni}$?

The answer, of course, is no. We specifically required the posets in algebraic stratifications be finite, and so any time you have an algebraic stratification, you can always just add a point as a new stratum. Since there are infinitely many points in X, we lose.

On the other hand, the answer can be made to be yes if we allow ourselves to relax the finiteness. But we shouldn't relax the finiteness by saying, 'ok, any size poset will do.' The way we have to relax the finiteness is by taking some *limit* of finite posets, because we want something that will map to every finite poset with which we can stratify $\mathbf{C}^n$. What we really want is to define P^{uni} as the limit of the posets P over all algebraic stratifications $X \to P$. So an element v of P^{uni} is a compatible system $\{v_p\}_{p\colon X \to P}$ of elements $v_p \in P$, one for every algebraic stratification $p\colon X \to P$. As the stratifications p get finer and finer, the strata X_{v_p} get smaller and smaller.

If you take a point $x \in X$, then you certainly get such a compatible system: namely, you can take $\{p(x)\}_{p\colon X \to P}$. But there are others!

In $\mathbf{C}^2$, consider the system $\{v_p\}_{p\colon X \to P}$ where $v_p \in P$ is the smallest element such that every point of the form (x, x^2) is contained in $X_{\leq v_p}$. Such a smallest element exists because algebraic stratifications can't break up the locus $\{y = x^2\}$ in a meaningful way: if you have an algebraic stratification $p\colon X \to P$, and if v is a minimal element such that $\{y = x^2\}$ is contained in $X_{\leq v}$, then the irreducibility of $y - x^2$ implies that all but finitely many points of the form (x, x^2) map to v.

This gives us a way to think about what's happening with the Zariski topological space $(\operatorname{Spec} R)_{zar}$ more generally. It turns out that this topological space comes from an *inverse system* of finite posets $\{P_\alpha\}_{\alpha \in \Lambda}$. Here, Λ is itself a poset such that if $\alpha, \beta \in \Lambda$, then there exists $\gamma \in \Lambda$ such that $\gamma \leq \alpha$ and $\gamma \leq \beta$. (One sometimes calls Λ a *cofiltered* poset.) The inverse system consists of the following:

- for each $\alpha \in \Lambda$ you have a finite poset P_α;

- if $\alpha \leq \beta$, you have a monotonic map $P_\alpha \to P_\beta$; and
- these are compatible with composition in the way you're already imagining.

In our case, the posets P_α are going to be *algebraic stratifications* of $\operatorname{Spec} R$. We can define these without making explicit reference to points of $\operatorname{Spec} R$, but let's postpone that story for now. Instead, I want to focus on the meaning of the topology.

The point is, once we've identified the correct notion of algebraic stratification, the Zariski topology is just trying to be the universal algebraic stratification. That isn't a finite poset itself (usually), but it is a limit of this inverse system of finite posets, and *that's* how the topology arises.

That is, you think of the Alexandrov topology on the posets P_α, and you form the limit in the category of topological spaces. The result is the Zariski topology. This is an important way to use topology that doesn't get the attention it deserves in undergrad textbooks: you start with some finite combinatorial gadgets that you give a ridiculously simple topology, and then you take infinite limits of these things in topological spaces. This carries us from gadgets to what are sometimes called *pro*-gadgets.

In fact, you may have encountered this kind of thing before. If you take an inverse system of finite sets (so posets where the only comparable elements are equal), then you can endow each of these with the discrete topology. Then when you take the limit of this system in topological spaces, you get what's called a *Stone space* – a compact, Hausdorff, totally disconnected topological space. Cantor spaces are a nice example, but there are others, like the Stone–Čech compactification of an infinite set. [5] In brief, profinite sets are the same thing as Stone spaces.

It's no big surprise that the topologies that come out of this sort of construction are 'weird' in the sense that they don't have much in common with Euclidean space. The topology is serving a different role than we're accustomed to: it's really only there to help us cope with the fact that we took an infinite limit of some combinatorial structures.

What we're witnessing is the phenomenon that there are really *two* ways to pass from finite mathematical objects of some kind to infinite mathematical objects of the same kind. On one hand, you can take a sequence of maps of finite gadgets

$$A_0 \to A_1 \to A_2 \to \cdots$$

and take the union (or colimit) $A_{+\infty}$. The result is an 'ind-object', which is what we tend to think of first when we think of passing from finite to infinite. It's easy to map out of an ind-object: a map out of $A_{+\infty}$ is a compatible system of maps out of each A_i. In other words, $A_{+\infty}$ is well approximated by the finite objects A_i from the left. For instance, if the A_i are all finite sets, then $A_{+\infty}$ is just an infinite set. If the A_i are all finite groups, then $A_{+\infty}$ is an infinite torsion group. If the A_i are all finite posets, then $A_{+\infty}$ is an infinite poset. This is the way we're used to passing from finite to infinite structures.

On the other hand, you can take a sequence going the other way:

$$\cdots \to A_{-2} \to A_{-1} \to A_0$$

and take the limit $A_{-\infty}$. This process gives you an 'pro-object', which is also infinite, but in a dual manner. It's easy to map *into* a pro-object: a map into $A_{-\infty}$ is a compatible system of maps into each A_i. In other words, $A_{-\infty}$ is well approximated by the finite objects A_i from the right. That means that pro-objects are infinite in a different way – one that is nicely modelled with topology. The resulting sets usually have big cardinalities (uncountable, typically), but as topological spaces they often have good compactness properties (by Tychonoff). For instance, if the A_i are all finite sets, then as we have seen, $A_{-\infty}$ is a Stone space. If the A_i are all finite groups, then $A_{-\infty}$ is a profinite group (i.e., a compact, Hausdorff, totally disconnected topological group).

If the A_i are all finite posets, then $A_{-\infty}$ is a *spectral topological space*. This is a topological space with the following properties:

- it is quasicompact;
- it is sober – *i.e.*, every irreducible closed subset contains a unique generic point;
- finite intersections of quasicompact subsets are quasicompact;
- the quasicompact opens form a base for the topology.

In brief, profinite posets are the same thing as spectral spaces. Melvin Hochster showed in the 1960s that spectral spaces are exactly [6] the topological spaces of the form $(\operatorname{Spec} R)_{zar}$.

The upshot here is that the Zariski topology is really just a *profinite poset*, and it has *exactly the same information* as the collection of all algebraic stratifications of $\operatorname{Spec} R$.

Once you take on this attitude toward the Zariski topology, it becomes a lot easier to accept. You can't ask too much of it: it merely assembles an infinite family of combinatorial structures.

In a later post, we'll explain our notion of algebraic stratification, and we'll describe the structure sheaf.

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1. One decision that was easy to make is that we'll use Vakil's text. It's just so good. It doesn't do things the same way I would, but that's a *good* thing. As James Richardson says, 'Shadows are harshest when there is only one lamp.'↩
2. An important principle that isn't often made explicit in undergraduate mathematics classes: full subcategories are cut out by properties; any other sort of forgetful functor reflects the presence of *structure*.↩
3. Well, of course we can. But the point is we can do it in a way that preserves all the information of a stratification.↩
4. Here's a teaser if you've never thought about this: what's the homotopy type of the nerve of the poset P?↩
5. If S is a set, the Stone–Čech compactification is the limit of all the finite sets to which S maps. It's also the set of ultrafilters on S.↩
6. Given a spectral space X, there's nothing unique about the rings R such that $X = \operatorname{Spec} R$. There are some constructions of choices of R that are 'nice', but one shouldn't be misled into thinking that a ring R can be recovered from the Zariski topology on its Spec.↩