Inductive limit topology on $C_c(X)$

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details to be checked…

Definition 1. *Let E be a vector space, and (E_α) be a family of topological vector spaces and for each α, let f_a be a linear mapping of E_α in E. The* inductive limit topology *on E with respect to the family (E_α, f_α) is the final locally convex topology on E such that each f_α is continuous.*

Proposition 2. *The final locally convex topology on E exists. Moreover, if $\mathcal{N}_\alpha$ be a base of neighborhoods of 0 in E_α for each α, then the following sets form a base of neighborhoods of 0 in E:*

$$\bigcup \{S \subset E \colon f_\alpha^{-1}(S) \in \mathcal{N}_\alpha \quad \forall \alpha\}.$$

Let X be a locally compact Hausdorff space. The inductive limit topology on $C_c(X)$ is with respect to the family $(C_c(X,K), i_K)$, where K runs over all compact subsets of X and i_K is the inclusion map of $C_c(X,K)$ in $C_c(X)$.

Proposition 3. *The following set is a base of inductive limit topology on $C_c(X)$:*

$$\bigcup_{K \subset X \text{ compact}} \{f \in C_c(X,K) \colon \|f\|_\infty < \frac{1}{n_K}\},$$

where n_K runs over all positive integers for each compact subset K of X.

Proposition 4. *The inductive limit topology, the compact convergence topology and the uniform topology coincide on $C_c(X,K)$ for each compact subset K of X.*

Proof. As the supremum norm topology on $C_c(X)$ is a convex topology such that each i_K is continuous, the inductive limit topology is finer than supremum norm topology on $C_c(X)$, and thus the restriction of inductive limit topology on $C_c(X,K)$ is finer than the supremum norm topology on $C_c(X,K)$.

For any open set V in the inductive limit topology on $C_c(X)$, $V \cap C_c(X,K) = i_K^{-1}(V)$ is an open set in the supremum topology on $C_c(X,K)$, i.e., the restriction of inductive limit topology on $C_c(X,K)$ is coarser than the supremum norm topology on $C_c(X,K)$. $\square$

Reference: Bourbaki, Topological Vector Spaces, II.29