A vector lattice norm is alwalys order continuous on $C_c(X)$

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Suppose X is a σ-compact Hausdorff space, μ is a regular Borel measure on X, and $E \subset L^0(X,\mu)$ is a Banach function space containing $C_c(X)$ as a sublattice.

By μ_τ denote the unique regular Borel measure such that

$$\tau(f) = \int_X f \, d\mu_\tau \tag{1}$$

for all $f \in C_c(X)$.

Lemma.
If $C_c(X)$ is dense in E, then TFAE.

(1) E is order continuous.

(2) For each positive function τ on E,

$$\mu(S) = 0 \text{ implies } \mu_\tau(S) = 0$$

for any Borel set S.

(2) $\Rightarrow$ (1).
Let f be an element in E_+ and (f_n) be a sequence in $C_c(X)_+$ convergent to f with respect to the norm on E, then $\tau(f_n)$ converges to $\tau(f)$.
Assume that f_n converges to f in order (replaced by a subsequence otherwise), then there is an increasing positive sequence (g_n) and a decreasing positive sequence (h_n) in E that are both convergent to f in order and satisfy $g_n \leq f_n \leq h_n$.

Since E is an order ideal of $L^0(X,\mu)$, g_n converges to f with respect to μ and thus with respect to μ_τ by the assumption in (2).
Therefore,

$$\int_X f \, d\mu_\tau = \lim_n \int_X g_n \, d\mu_\tau \leq \lim_n \int_X f_n \, d\mu_\tau = \lim_n \tau(f_n) = \tau(f).$$

By the arbitrariness of f,

$$E \subset L^1(\mu_\tau).$$

Now we have

$$\int_X f \, d\mu_\tau = \lim_n \int_X h_n \, d\mu_\tau \geq \lim_n \int_X f_n \, d\mu_\tau = \tau(f),$$

hence the equation (1) holds for all $f \in E$.

Suppose (e_n) is a monotone order bounded sequence in E, then e_n increases or decreases to some e in E almost everywhere with respect to μ_τ.
Thanks to Dini's Theorem, e_n converges to e with respect to the norm on E, i.e., E is order continuous.

(1) $\Rightarrow$ (2).
Suppose $\mu(S) = 0$, K is a compact subset of S, and $\tau \in E_+^*$.
Let (O_n) be a sequence of open sets containing K such that $(\mu + \mu_\tau)(O_n - K)$ converges to 0. We can find a decreasing sequence (f_n) in $C_c(X)$ such that $1_K \le f_n \le 1_{O_n}$ for all n.
Replacing with this sequence, we can assume f_n decreases to 1_K almost everywhere with respect to $\mu + \mu_\tau$. Hence, f_n decreases to $1_K = 0$ in order.
Since E is order continuous, f_n converges to 0 weakly, that is, $\tau(f_n)$ converges to 0.
However, f_n decreases to 1_K almost everywhere with respect to μ_τ, which implies that $\tau(f_n) = \int_X f_n \, d\mu_\tau$ converges to $\mu_\tau(K) = \int_X 1_K \, d\mu_\tau$, so $\mu_\tau(K) = 0$. As X is σ-compact, $\mu_\tau(S) = \sup\{\mu_\tau(K) \mid K \subset S\} = 0$.

Ignore the old version.

Theorem. Suppose X is a locally compact space, E is a Dedekind complete Banach lattice which consists of Borel measurable functions on X and contains $C_c(X)$ as a subspace.

(a) For any compact subset K of X, the inclusion map $i : (C(K), \|\cdot\|_\infty) \to E$ is continuous.

(b) For any decreasing sequence (f_n) in $C_c(X)$, f_n converges to 0 with respect to order is equivalent to f_n converges to 0 with respect to the norm on E.

(c) The norm on E is order continuous if $C_c(X)$ is dense in E.

Proof.
(a) For each $f \in C(K)$,

$$\|i(f)\| \le \|i(\|f\|_\infty 1_K)\| = \|i(1_K)\| \|f\|_\infty.$$

(b) Let $K = \operatorname{supp} f_1$, then $f_n \in C(K)$ for all n. Suppose τ is a positive linear functional on E, then $\tau \circ i$ is a positive linear functional on $C(K)$ where i is the inclusion map from $(C(K), \|\cdot\|_\infty)$ to E. Since there is a unique regular Borel measure μ on X such that $\int f \, d\mu = \tau \circ i(f) \, (f \in C(K))$, $\tau \circ i(f_n)$ converges to 0 by the Dominated Convergence Theorem. Hence, $i(f_n)$ $\sigma(E, E^*)$-converges to 0. By Dini's Theorem, $f_n = i(f_n)$ converges to 0 with respect to the norm on E.

(c) Suppose (g_n) is a decreasing sequence in E and (f_n) is a sequence in $C_c(X)$ such that $\|f_n - g_n\| \le \frac{1}{2^n}$. Let $h_n := f_1^+ \wedge f_2^+ \wedge \cdots \wedge f_n^+$, then h_n is a decreasing sequence in $C_c(X)_+$ such that $\|h_n - g_n\| \leq \frac{1}{2^{n-1}}$. So, there is a subsequence and a positive element $u \in E$ such that $|h_{k_n} - g_{k_n}| \leq \frac{1}{n}u$. Since $0 \leq h_{k_n} \leq \frac{1}{n}u + g_{k_n} \downarrow 0$, (h_{n_k}) converges to 0 in order and so does (h_n). Hence (h_n) converges to 0 in norm by (a). The proof ends with (g_n) converging to 0 in norm.