

A vector lattice norm is always order continuous on $C_c(X)$

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Suppose X is a σ -compact Hausdorff space, μ is a regular Borel measure on X , and $E \subset L^0(X, \mu)$ is a Banach function space containing $C_c(X)$ as a sublattice.

By μ_τ denote the unique regular Borel measure such that

$$\tau(f) = \int_X f d\mu_\tau \quad (1)$$

for all $f \in C_c(X)$.

Lemma.

If $C_c(X)$ is dense in E , then TFAE.

1. E is order continuous.
2. For each positive function τ on E ,

$$\mu(S) = 0 \text{ implies } \mu_\tau(S) = 0$$

for any Borel set S .

(2) \Rightarrow (1).

Let f be an element in E_+ and (f_n) be a sequence in $C_c(X)_+$ convergent to f with respect to the norm on E , then $\tau(f_n)$ converges to $\tau(f)$.

Assume that f_n converges to f in order (replaced by a subsequence otherwise), then there is an increasing positive sequence (g_n) and a decreasing positive sequence (h_n) in E that are both convergent to f in order and satisfy

$$g_n \leq f_n \leq h_n.$$

Since E is an order ideal of $L^0(X, \mu)$, g_n converges to f with respect to μ and thus with respect to μ_τ by the assumption in (2).

Therefore,

$$\int_X f d\mu_\tau = \lim_n \int_X g_n d\mu_\tau \leq \lim_n \int_X f_n d\mu_\tau = \lim_n \tau(f_n) = \tau(f).$$

By the arbitrariness of f ,

$$E \subset L^1(\mu_\tau).$$

Now we have

$$\int_X f d\mu_\tau = \lim_n \int_X h_n d\mu_\tau \geq \lim_n \int_X f_n d\mu_\tau = \tau(f),$$

hence the equation (1) holds for all $f \in E$.

Suppose (e_n) is a monotone order bounded sequence in E , then e_n increases or decreases to some e in E almost everywhere with respect to μ_τ .

Thanks to Dini's Theorem, e_n converges to e with respect to the norm on E , i.e., E is order continuous.

(1) \Rightarrow (2).

Suppose $\mu(S) = 0$, K is a compact subset of S , and $\tau \in E_+^*$.

Let (O_n) be a sequence of open sets containing K such that $(\mu + \mu_\tau)(O_n - K)$ converges to 0. We can find a decreasing sequence (f_n) in $C_c(X)$ such that $1_K \leq f_n \leq 1_{O_n}$ for all n .

Replacing with this sequence, we can assume f_n decreases to 1_K almost everywhere with respect to $\mu + \mu_\tau$. Hence, f_n decreases to $1_K = 0$ in order. Since E is order continuous, f_n converges to 0 weakly, that is, $\tau(f_n)$ converges to 0.

However, f_n decreases to 1_K almost everywhere with respect to μ_τ , which implies that $\tau(f_n) = \int_X f_n d\mu_\tau$ converges to $\mu_\tau(K) = \int_X 1_K d\mu_\tau$, so $\mu_\tau(K) = 0$. As X is σ -compact, $\mu_\tau(S) = \sup\{\mu_\tau(K) \mid K \subset S\} = 0$.

Ignore the old version.

Theorem. Suppose X is a locally compact space, E is a Dedekind complete Banach lattice which consists of Borel measurable functions on X and contains $C_c(X)$ as a subspace.

- a. For any compact subset K of X , the inclusion map $i : (C(K), \|\cdot\|_\infty) \rightarrow E$ is continuous.
- b. For any decreasing sequence (f_n) in $C_c(X)$, f_n converges to 0 with respect to order is equivalent to f_n converges to 0 with respect to the norm on E .
- c. The norm on E is order continuous if $C_c(X)$ is dense in E .

Proof.

(a) For each $f \in C(K)$,

$$\|i(f)\| \leq \|i(\|f\|_\infty 1_K)\| = \|i(1_K)\| \|f\|_\infty.$$

- b. Let $K = \text{supp } f_1$, then $f_n \in C(K)$ for all n . Suppose τ is a positive linear functional on E , then $\tau \circ i$ is a positive linear functional on $C(K)$ where i is the inclusion map from $(C(K), \|\cdot\|_\infty)$ to E . Since there is a unique regular Borel measure μ on X such that

$\int f d\mu = \tau \circ i(f)$ ($f \in C(K)$), $\tau \circ i(f_n)$ converges to 0 by the Dominated Convergence Theorem. Hence, $i(f_n)$ $\sigma(E, E^*)$ -converges to 0. By Dini's Theorem, $f_n = i(f_n)$ converges to 0 with respect to the norm on E .

- c. Suppose (g_n) is a decreasing sequence in E and (f_n) is a sequence in $C_c(X)$ such that $\|f_n - g_n\| \leq \frac{1}{2^n}$. Let $h_n := f_1^+ \wedge f_2^+ \wedge \cdots \wedge f_n^+$, then h_n is a decreasing sequence in $C_c(X)_+$ such that $\|h_n - g_n\| \leq \frac{1}{2^{n-1}}$. So, there is a subsequence and a positive element $u \in E$ such that $|h_{k_n} - g_{k_n}| \leq \frac{1}{n}u$. Since $0 \leq h_{k_n} \leq \frac{1}{n}u + g_{k_n} \downarrow 0$, (h_{k_n}) converges to 0 in order and so does (h_n) . Hence (h_n) converges to 0 in norm by (a). The proof ends with (g_n) converging to 0 in norm.