Order convergence in an order ideal

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A net $(x_i)_{i\in I}$ in a vector lattice is said to be *order convergent* to x if there is a decreasing net $(y_j)_{j\in J}$ with infimum 0 such that for any $j \in J$ there is some $i_0 \in I$ such that $|x_i - x| \leq y_j$ for all $i \geq i_0$.

The following lemma should exist in the literature, but we are not aware of a reference.

Lemma 1. *Let E be a vector lattice, let F be an order ideal of E, let $(x_i)_{i\in I}$ be a net in F, and let $x \in F$. Then the net $(x_i)_{i\in I}$ is order convergent to x in F if and only if it is order convergent to x in E and the absolute value of one of its tails is dominated by a positive element of F.*

Proof. Suppose that $(x_i)_{i\in I}$ is order convergent to x in F. There is a decreasing net $(y_j)_{j\in J}$ with infimum 0 in F such that, for any $j \in J$, there is some $i_0 \in I$ such that $|x_i - x| \leq y_j$ for all $i \geq i_0$. Clearly, 0 is a lower bound for $(y_j)_{j\in J}$ in E. Suppose that $y \in E$ is also a lower bound. Since then $0 \leq y \vee 0 \leq y_j$ for all $j \in J$, we have that $y \vee 0 \in F$ by the definition of order ideal, and then also that $y \vee 0 \leq 0$. Hence $y \leq 0$, so that 0 is the infimum of $(y_j)_{j\in J}$ in F. We see that $(x_i)_{i\in I}$ is order convergent to x in E. With j_0 and i_0 as above, we have that

$$|x_i| \leq |x_i - x| + |x| \leq y_{j_0} + |x|$$

for $i \geq i_0$. Hence the tail $(|x_i|)_{i\geq i_0}$ is bounded by $y_{j_0} + |x|$.

Conversely, suppose that $(x_i)_{i\in I}$ is order convergent to x in E and such that the absolute value of one of its tails is dominated by a positive element y of F. It is sufficient to prove that such a tail is order convergent to x in F, so we may as well suppose that $|x_i| \leq y$ for all i. Let $(y_j)_{j\in J}$ be a decreasing net in E with infimum 0 in E such that for any $j \in J$ there is some $i_0 \in I$ such that $|x_i - x| \leq y_j$ for all $i \geq i_0$. Set $y'_j \coloneqq y_j \wedge (y + |x|)$, then (y'_j) is a decreasing net with infimum 0 in F and $|x_i - x| \leq y'_j$ for all $i \geq i_0$. Hence $(x_i)_{i\in I}$ is order convergent to x in F. □

Proposition 2. *Let E be a vector lattice, I a directed set, $(x_i)_{i\in I}$ a net in E and $x \in E$. (x_i) converges to x if and only if there are two nets $(y_j)_{j\in J}$ and $(z_j)_{j\in J}$ in E such that y_j decreasing to x, z_j increasing to x and $y_j \leq x_i \leq z_j$ for all $j \in J$, where J is a directed set that is cofinal in I. If I is the set $\mathbb{N}$ of natural numbers, the we can also take $J = \mathbb{N}$.*

The statement can be easily verified by definition but is conventional for usage.