(almost) interval preserving maps preserve order continuity

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Five months ago, Marcel de Jeu and I published an article titled
Direct limits in categories of normed vector lattices and Banach lattices.
The paper is written in a self-contained manner, covering many known results so that readers don't need to consult external references while reading.

This blog post aims to highlight what is new in the paper:

- Introduction of direct limits to the category of Banach lattices with contractive (almost) interval-preserving maps, say **C**.
- A counterexample showing that the canonical construction of direct limits does not work in the category **C**.
- A proof that order continuity is an invariant property under direct limits in **C**.
- A result: the density of $C_c(X)$ in a Banach function space implies order continuity of the space, where X is a locally compact Hausdorff space whose compact subsets are all metrizable.

Quick review of definitions

A *Banach lattice* is a Banach space with an lattice order such that such that for all $x, y \in X$, the implication

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

holds, where the absolute value $|\cdot|$ is defined as $|x| = x \vee (-x)$. A *Banach function space* is a Banach lattice consists of measurable functions on a measure space. Canonical examples include $L^p(X)$ and $C_0(X)$ where X is a locally compact Hausdorff space.

We call a Banach lattice *order continuous* if every monotone order bounded sequence is convergent in norm.

A linear map $\varphi\colon E \to F$ between Banach lattices is said to be *interval preserving* if it is positive and such that $\varphi([0,x]_E) = [0,\varphi(x)]_F$ for all positive x in E.

A linear map $\varphi\colon E \to F$ between Banach lattices is called *almost interval preserving* if it is positive and such that $[0,\varphi(x)]_F = \overline{\varphi([0,x]_E)}$ for all all positive x in E.