Dye's theorem for C*-algebras

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The following is a main result in the paper joint with Chi-Keung Ng Ortho-sets and Gelfand spectra .

Definition 0.1 *Let $\mathcal{L}$ be an ortholattice and $p, q \in \mathcal{L}$, and denote by p' the orthocomplement of p.*
(a) We say that p q-commutes *with q if $p \wedge (p \wedge q)' \leq (q \wedge (p \wedge q)')'$.*
(b) $p \in \mathcal{L}$ is said to be q-central *if it q-commutes with all other elements in $\mathcal{L}$.*

Definition 0.2 *Let $\mathcal{L}$ be an ortholattice. A* quantum topology *on $\mathcal{L}$ is a subcollection $\mathcal{C} \subseteq \mathcal{L}$ satisfying:*

S1). $0, 1 \in \mathcal{C}$;

S2). if $\{p_\lambda\}_{\lambda\in\Lambda}$ is a family in $\mathcal{C}$, then $\bigwedge_{\lambda\in\Lambda} p_\lambda$ exists and belongs to $\mathcal{C}$;

S3). if p and q are q-commuting elements in $\mathcal{C}$ then $p \vee q \in \mathcal{C}$.

In this case, elements in $\mathcal{C}$ are said to be quantum closed*, while elements of the form p' for some $p \in \mathcal{C}$ are said to be* quantum open.

For a quantum system modeled on the self-adjoint part $B_{\rm sa}$ of a (complex) C^*-algebra B, we introduce the semi-classical object of Gelfand spectrum for this system as follows. Let $P(B)$ be the set of all pure states on B. For any $\phi, \psi \in P(B)$, we denote $\phi \neq_{\rm o} \psi$ if ϕ and ψ have orthogonal support projections; i.e., ϕ and ψ has zero transition probability. For any left closed ideal $L \subseteq B$, we set

$$\begin{aligned}\mathrm{hull}(L) &:= \{\phi \in P(B) : \phi(x^*x) = 0,\ \text{for every } x \in L\} \\ &= \{\phi \in P(B) : L \subseteq \ker\phi\}\end{aligned}$$

(see e.g., (Murphy 1990, Theorem 5.3.4)). Then $\mathrm{hull}(L)$ is a q-subset of $(P(B), \neq_{\rm o})$, and the collection of all such q-subsets form a quantum topology $\mathcal{C}^B$ on $(P(B), \neq_{\rm o})$.

The good point for the Gelfand spectrum of a C^*-algebra is that it captures the self-adjoint part of original algebra up to a Jordan isomorphism (under a mild assumption), which is good enough for the consideration of physical structure modeled on the self-adjoint parts of C^*-algebras. Let us recall that a linear map Γ from the self-adjoint part of a C^*-algebra A to that of another C^*-algebra is a *Jordan isomorphism* if it preserves the Jordan product; i.e. $\Gamma(ab + ba) = \Gamma(a)\Gamma(b) + \Gamma(b)\Gamma(a)$ $(a, b \in A_{\mathrm{sa}})$. The following can be seen as a non-commutative generalization of the Gelfand theorem:

Theorem 1 *Let A and B be two C^*-algebras. Suppose that there is a bijection $\Psi : P(A) \to P(B)$ preserving the q-distinctness relations such that $\mathcal{C}^B = \left\{\Psi(C) : C \in \mathcal{C}^A\right\}$.*

(a) If A has no 2-dimensional irreducible *-representation, then there is a Jordan isomorphism $\Gamma : B_{\mathrm{sa}} \to A_{\mathrm{sa}}$ such that $\Psi(\omega) = \omega \circ \Gamma$ $(\omega \in P(A))$.

(b) If A is simple (including the case when $A = \mathbb{M}_2$), then A and B are either *-isomorphic or *-anti-isomorphic.

References

Murphy, GJ. 1990. *C* Algebras and Operator Theory*.