Solutions to some exercises in Murphy's *C*-algebra and Operator Theory*

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3.2. Let τ be a positive linear function on A.
(a)If I is a closed ideal in A, show that $I \subset \ker \tau$ iff $I \subset \ker \varphi_\tau$.
(b)We say τ is faithful if $\tau(a) = 0 \Rightarrow a = 0 \ \forall a \in A^+$. Show that if τ is faithful, then the GNS representation (H_τ, φ_τ) is faithful.
(c)Suppose that α is an automorphism of A such that $\tau(\alpha(a)) = \tau(a)$ for all $a \in A$. Define a unitary on H_τ by setting $u(a + N_\tau) = \alpha(a) + N_\tau(a \in A)$. Show that $\varphi_\tau(\alpha(a)) = u\varphi_\tau(a)u^*(a \in A)$.

Proof. (a)
$I \subset \ker \varphi_\tau \Leftrightarrow \varphi_\tau(a) = 0 \forall a \in I \Leftrightarrow \varphi(a) = 0 \forall a \in I \Leftrightarrow$ for all $a \in I$ and $b \in A, \varphi(a)(b + N_\tau) = 0 \Leftrightarrow \forall a \in I$ and $b \in A, ab \in N_\tau \Leftrightarrow \forall a \in I, b \in A$ and $c \in A, \tau(cab) = 0 \overset{I=I^3}{\Leftrightarrow} \tau(a) = 0 \forall a \in I \Leftrightarrow I \subset \ker \tau$.

(b)Let $I = \ker \varphi_\tau$, then I is a closed ideal in A. By A, $I \subset \ker \tau$. For all $a \in I$, $a^*a \in I$, so $\tau(a^*a) = 0$, hence $a^*a = 0$ and thus $a = 0$. Therefore, $\ker \varphi_\tau = 0$.

(c)u is a unitary is equal to u is a isomorphism in the sense u is a surjection which preserves inner product. Denote $a + N_\tau$ by $[a]$, then
$\langle u[a], u[b] \rangle = \langle [\alpha(a)], [\alpha(b)] \rangle = \tau(\alpha(b)^*\alpha(a)) = \tau(\alpha(b^*a)) = \tau(b^*a) = \langle [a], [b] \rangle$;

$\alpha(a) - a \in N_\tau, \forall a \in A$ thus u is a unitary.

$u\varphi_\tau(a)u^*([b]) = u\varphi_\tau(a)u^{-1}([b]) = u\varphi(a)$ □

3.3. If $\varphi : A \to B$ is a positive linear map between C^* algebras, show that φ is bounded.

Proof. For all positive linear functional τ on B, $\tau \circ \varphi$ is a positive linear functional on A and thus bounded, i.e.

$$\|\tau \circ \varphi(x)\| \leq M_\tau \quad (\forall x \in A \text{ satisfying } lx \leq 1)$$

for some positive constant M_τ.

Every bounded linear functional on B is sum of 4 positive linear functionals on B by Jordan Decomposition, thus for all $b^* \in B^*$,

$$lb^*(\varphi(x)) \leq M_{b^*} \quad (\forall x \in A \text{ satisfying } lx \leq 1)$$

for some positive constant M_{b^*}. Hence $l\varphi(x) \leq M$ for some positive constant M by the Principle of Uniform Boundedness, i.e. φ is bounded. □

3.5. An element a of A_+ is strictly positive if the hereditary C^*-subalgebra of A generated by a is A itself, that is, if $(aAa)^- = A$.
(a) Show that if A is unital, then $a \in A_+$ is strictly positive if and only if a is invertible.
(b) If H is a Hilbert space, show that a positive compac operator on H is strictly positive in $K(H)$ if and only if it has dense range.
(c) Show that if a is strictly positive in A, then $\tau(a) > 0$ for all non-zero positive linear functionals τ on A.

Proof. (a) If a is invertible, then $aAa = A$, thus a is strictly positive.

Conversely, if a is strictly positive, then $(aAa)^- = A$. Since the set G of all invertible elements is open, $aAa \cap G \neq \emptyset$. Hence there exists some $b \in A$ such that aba is invertible, thus a is invertible.

(b)If u is strictly positive in $K(H)$, that is $(uK(H)u)^- = K(H)$, then $\forall x \in H$, there exists a sequence $\{v_n\}$ in $B(H)$ such that

$$x \otimes x = \lim_{n\to\infty} uv_n u.$$

Hence

$$lx^2 x = x \otimes x(x) = \lim_{n\to\infty} u(v_n u(x)) \in (uH)^-.$$

Therefore, $x \in (uH)^-$ and $H = (uH)^-$.

Conversely, if $(uH)^- H$, then for all $x \in H$, then there exists a sequence $\{h_n\}$ in H such that $uh_n \to x$, hence

$$u(h_n \otimes h_n)u = (uh_n) \otimes (uh_n) \to x \otimes x.$$

Therefore, every rank-one projection belongs to $(uK(H)u)^-$. Since all the rank-one projections is the total subset of $K(H)$, $K(H) = (uK(H)u)^-$.

(c)Suppose τ is a positive functional such that $\tau(a) = 0$, then

$$\tau((a^{\frac{1}{2}})^* a^{\frac{1}{2}}) = 0,$$

so

$$\tau(a^* a) = \tau((a^{\frac{1}{2}})^3 a^{\frac{1}{2}}) = 0.$$

Hence for all $b \in A$,

$$\tau(aba) = 0.$$

Therefore, $\tau = 0$ for $A = (aAa)^-$. $\square$

4.4. Let A be a von Neumann algebra on a Hilbert space H, and suppose that τ is a bounded linear functional on A. We say that τ is normal if, whenever an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ in A_{sa} converges strongly to an operator $u \in A_{sa}$, we have $\lim_\lambda \tau(u_\lambda) = \tau(u)$. Show that every σ-weakly continuous functional $\tau \in A^*$ is normal.

Preliminaries 0.1. *Suppose u, v are two bounded linear operators on some Hilbert space, then*

$$|uv| \leq \|u\||v|.$$

Indded,Let $v = w|u|$ and $uv = w'|uv|$ be the Polar Decomposition, then $|uv|^2 = w'^* uw|v|$, thus

$$\begin{aligned} |uv|^2 &= |v|w^* u^* w' w'^* uw|v| \\ &\leq \|w'^* uw\|^2 |v|^2 \leq lu^2 |v|^2, \end{aligned}$$

Hence $|uv| \leq \|u\||v|$.

Theorem 0.2. *Let A be a von Neumann algebraon Hilbert space containing id_H, then for any σ-continuous linear τ on A, there exists $u \in L^1(H)$ such that*

$$\tau(v) = \mathrm{tr}(uv)$$

for all $v \in A$.

Proof. Suppose $\{e_\alpha\}_{\alpha \in \Gamma}$ is a basis for H, $u \in L^1(H)$ such taht $\tau(v) = \mathrm{tr}(uv)$ for all $v \in A$. and a bounded net $(v_\lambda)_{\lambda \in \Lambda}$ in A_{sa} converges strongly to an operator $v \in A_{sa}$. For every $\varepsilon > 0$, there exists a finite subset Γ_0 of Γ such that

$$\sum_{\alpha \notin \Gamma_0} \langle |u| e_\alpha, e_\alpha \rangle \leq \varepsilon,$$

and there exists $\lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0$

$$l(v_\lambda - v)ue_\alpha \leq \varepsilon, \alpha \in \Gamma_0.$$

Therefore,

$$\begin{aligned} l(v_\lambda - v)u_1 &= \sum_{\alpha \in \Gamma} \langle |(v_\lambda - v)u| e_\alpha, e_\alpha \rangle \\ &\leq \sum_{\alpha \notin \Gamma_0} \langle |u| e_\alpha, e_\alpha \rangle l v_\lambda - v + \sum_{\alpha \in \Gamma_0} l(v_\lambda - v)ue_\alpha \\ &\leq M\varepsilon \end{aligned}$$

for some constant M. Hence $|\tau(v_\lambda) - \tau(v)| \leq l(v_\lambda - v)u_1 \to 0$. □

4.6. Let A be a C*-algebra.
(a) Show that if A is unital, then its unit is an extreme point of its closed unit ball.
(b) If p is a projection of A, show that it is an extreme point of the closed unit ball of A^+.

4.7. Let A be a C^*-algebra.
(1)Show that if p, q are equivalent projections in A, and r is a projection orthogonal to both (that is, $rp = rq = 0$), then the projections $r + p$ and $r + q$ are equivalent.
(2)If H is a separable Hilbert space and p is a projection not of finite rank, set $\text{rank}(p) = \infty$. If p has finite rank, set $\text{rank}(p) = \dim p(H)$. Show that $r \sim q$ in $B(H)$ if and only if $\text{rank}(p) = \text{rank}(q)$. Thus, the equivalence class of a projection in a C*-algebra can be thought of as its "generalised rank."
(3)We say a projection p in a C^*-algebra A is finite if for any projection q such that $q \sim p$ and $q \leq p$ we necessarily have $q = p$. Otherwise, the projection is said to be infinite. Show that if p, q are projections such that $q \leq p$ and p is finite, then q is finite.
(4)A projection p in a von Neumann algebra A is abelian if the algebra pAp is abelian. Show that abelian projections are finite.
(5)A von Neumann algebra is said to be finite or infinite according as its unit is a finite or infinite projection. If H is a Hilbert space, show that the von Neumann algebra $B(H)$ is finite or infinite according as H is finite- or infinite- dimensional.

Proof. (1) Suppose $p = u^*u$ and $q = uu^*$, then $ru^*ur = rpr = 0$ and $ruu^*r = rqr = 0$, so $ur = ru^* = ru = u^*r = 0$.Hence

$$r + p = r^2 + u^*u + u^*r + ru = (r + u^*)(r + u);$$
$$r + q = r^2 + uu^* + ur + ru^* = (r + u)(r + u^*),$$

that is, $r + p \sim r + q$.

(2)If $\text{ran}(p)$ and $\text{ran}(q)$ have a same dimension less than $\aleph_0$, choose orthonormal bases $\{e_n\}_{n=1}^N$ and

$\{f_n\}_{n=1}^\infty$ for $p(H)$ and $q(H)$, respectively(N may be infty). Let

$$u : \text{ran}(p) \to \text{ran}(q), u(e_n) = f_n, n = 1, 2, \cdots, N.$$

and let

$$v(x) = \begin{cases} u(x), x \in \text{ran}(p) = (\ker(p))^\perp; \\ 0, x \in \ker(p), \end{cases}$$

then $v^*v = p$(because they coincide on $\ker(p)$ and $(\ker(p))^\perp$) and $vv^* = q$.

Conversely, if $p \sim q$, then $p = u^*u$ and $q = uu^*$ for some $u \in B(H)$, so

$$\text{rank}(p) = \text{rank}(u^*qu) \leq \text{rank}(q);$$
$$\text{rank}(q) = \text{rank}(upu^*) \leq \text{rank}(p).$$

(3)Since $(p - q)r = 0, (p - q)q = 0$, $r + (p - q) \sim q + (p - q) = p$ by (1). Because $r + (p - q) \leq p$, $r + (p - q) = p$, i.e. $r = q$.

(4)Since pAp is an abelian C^*-algebra with the identity p, there is a $*$-isomorphism $\varphi : pAp \to C(\Omega)$, where Ω is the compact set of all non-zero algebra homomorphism from pAp to $\mathbb{C}$.

Suppose $q \leq p$, $p = u^*u$ and $q = uu^*$, then

$$\begin{aligned} u = qu &\xlongequal{q=pq} pqu \\ &\xlongequal{u=up} pqup \in pAp, \end{aligned}$$

thus

$$\begin{aligned} p = \varphi \ni (\varphi(p)) = \varphi \ni (\varphi(u)^*\varphi(u)) = \varphi \ni (|\varphi(u)|^2); \\ q = \varphi \ni (\varphi(q)) = \varphi \ni (\varphi(u)\varphi(u)^*) = \varphi \ni (|\varphi(u)|^2), \end{aligned}$$

so $p = q$.

(5)If H is finite-demensional, then $\operatorname{rank}(1) < \infty$, so 1 is finite, i.e., $B(H)$ is finite.

If H is infinite-demensional, suppose $\{e_n\}_{n=0}^{\infty} \sqcup \{e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for H, and set

$$u(e_n) = e_{n+1}(n = 0, 1, \cdots), u(e_\lambda) = e_\lambda(\lambda \in \Lambda).$$

Then

$$u^*(e_n) = e_{n-1}(n = 1, 2, \cdots),$$

$$u^*(e_0) = 0, u^*(e_\lambda) = e_\lambda.$$

Thus

$$u^*u(e_n) = e_n(n = 0, 1, \cdots),$$

$$uu^*(e_n) = e_n(n = 1, 2, \cdots), uu^*(e_0) = 0.$$

Let $p = uu^*$, then $p \sim u^*u = 1$, and $p \leq 1$, but $p \neq 1$, that is 1 is infinite, so is the Hilbert space. □

5.1. Let τ be a pure state on a C*-algebra A, and y a unit vector in H_τ such that $\tau(a) = \langle \varphi_\tau(a)y, y\rangle$ for all $a \in A$. Show that there is a scalar λ of modulus one such that $y = \lambda x$.

5.2. Let H be a Hilbert space and x a unit vector of H. Show that the functional $\omega_x : B(H) \to \mathbb{C}, u \mapsto \langle u(x), x\rangle$ is a pure state of $B(H)$.

5.3. Give an example to show that a quotient C*-algebra of a primitive C*-algebra need not be primitive.

5.4. If I is a primitive ideal of a C*-algebra A, show that $M_n(I)$ is a primitive ideal of $M_n(A)$. (Thus, if A is primitive, so is $M_n(A)$.)

5.5. Let A be a C*-algebra. Show the following conditions are equivalent:
(a) A is prime.
(b) If $aAb = 0$, then a or $b = 0$ $(a, b \in A)$.

5.6. Let $\mathcal{S}$ be a set of C*-subalgebras of a C*-algebra A that is upwardsdirected, that is, if $B, C \in \mathcal{S}$, then there exists $D \in \mathcal{S}$ such that $B, C \subset D$. Show that $(\cup\mathcal{S})^-$ is a C*-subalgebra of A.
Suppose that all the algebras in $\mathcal{S}$ are prime and that $A = (\cup\mathcal{S})^-$. Show that A is prime.

5.7. If A is a C*-algebra, its centre C is the set of elements of A commuting with every element of A. Show that C is a C*-subalgebra of A. Show that if $A$is simple, then $C = 0$ if A is non-unital and $C = \mathbb{C}1$ if A is unital.

5.8. Let $\mathcal{S}$ be an upwards-directed set of closed ideals in a C*-algebra A (cf. Exercise 5.6 for the term upwards-directed). Suppose that $A = (\cup\mathcal{S})^-$, and that all of the algebras in $\mathcal{S}$ are postliminal. Show that A is postliminal.

5.9. Let A be a C*-algebra. If I, J are postliminal ideals in A (that is, closed ideals that are postliminal C*-algebras), show that $I + J$ is postliminal also. Deduce from this and Exercise 5.8 that there is a largest postliminal ideal I in A (which may, of course, be the zero ideal). Show that A/I has no non-zero postliminal ideals.

6.1. Let $(A_n, \varphi_n)_{n=1}^\infty$ and $(B_n, \psi_n)_{n=1}^\infty$ be direct sequences of C*-algebras with direct limits A and B, respectively. Let $\varphi^n : A \to A$ and $\psi^n : B_n \to B$ be the natural maps. Suppose there are *-homomorphisms $\pi : A_n \to B_n$ such that for each n the following diagram commutes:

$$\begin{tikzcd} A_n \arrow[r, "\varphi_n"] \arrow[d, "\pi_n"'] & A_{n+1} \arrow[d, "\pi_{n+1}"] \\ B_n \arrow[r, "\psi_n"] & B_{n+1} \end{tikzcd}$$

Show that there exists a unique *-homomorphism $\pi : A \to B$ such that for each n the following diagram commutes:

$$\begin{tikzcd} A_n \arrow[r, "\varphi^n"] \arrow[d, "\pi_n"'] & A \arrow[d, "\pi"] \\ B_n \arrow[r, "\psi^n"] & B \end{tikzcd}$$

Show that if all the π_n are *-isomorphisms, then π is a *-isomorphism.

6.2. Show that every non-zero finite-dimensional C*-algebra admits a faithful tracial state. Give an example of a unital simple C*-algebra not having a tracial state.

6.3. Let A be a C*-algebra. A *trace* on A is a function $\tau : A^+ \to [0, \infty]$ such that

$$\begin{aligned} \tau(a+b) &= \tau(a) + tau(b) \\ \tau(ta) &= t\tau(a) \\ \tau(C^*c) &= \tau(cc^*) \end{aligned}$$

for all $a, b \in A^+, c \in A$, and all $t \in \mathbb{R}$. We use the convention that $0 \cdot \infty = 0$. The motivating example is the usual trace function on $B(H)$. Another example is got on $C_0(\mathbb{R})$ by setting $\tau(f) = \int f \mathrm{d}m$ where $f \in C_0(\mathbb{R}^+)$ and m is ordinary Lebesgue measure on $\mathbb{R}$.
Traces (and their generalisation, weights) play a fundamental role, especially in von Neumann algebra theory ([Ped], [Tak]).
Let

$$A = \{a \in A | \tau(a^*a) < \infty\}.$$

Show that

$$(a+b)^*(a+b) \leq 2a^*a + 2b^*b$$

and

$$(ab)^*ab \leq \|a\|^2 b^*b,$$

and deduce that A_τ^2 is a self-adjoint ideal of A.
Show that for arbitrary $a, b \in A$,

$$a^*b = \frac{1}{4}\sum_{k=0}^{3} i^k (b + i^k a)^*(b + i^k a),$$

and ......